

Precalculus Workbook

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A Companion to *Functions Modeling Change*
Connally, Hughes-Hallett, Gleason, et. al.

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Note to Students (Please Read): This workbook contains examples and exercises that will be referred to regularly during class. Please purchase or print out the rest of the workbook before our next class and bring it to class with you every day.

1. **To Print Out the Workbook.** Go to the web address below

<http://www.sonoma.edu/users/m/morrisj/m160/frame.html>

and click on the link “Math 160 Workbook”, which will open the workbook as a .pdf file. BE FOREWARNED THAT THERE ARE LOTS OF PICTURES AND MATH FONTS IN THE WORKBOOK, SO SOME PRINTERS MAY NOT ACCURATELY PRINT PORTIONS OF THE WORKBOOK. If you do choose to try to print it, please leave yourself enough time to purchase the workbook before our next class in case your printing attempt is unsuccessful.

2. **To Purchase the Workbook.** Go to *Digi-Type*, the print shop at 1726 E. Cotati Avenue (across from campus, in the strip mall behind the Seven-Eleven). Ask for the workbook for Math 160 - Precalculus. The copying charge will probably be between \$10.00 and \$20.00. You can also visit the *Digi-Type* webpage (<http://www.digi-type.com/>) to order your workbook ahead of time for pick-up.

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Chapter 1 Tools – Points and Linear Equations

Example 1. Solve $\frac{2}{2-t} = \frac{3}{2-2t}$ for t .

Example 2. Solve $2 = \frac{A+Bt}{A-Bt}$ for t .

Example 3. Write $\frac{A}{C+Bt} + \frac{B}{D-Ct}$ as a single fraction.

Example 4. Solve $\begin{cases} 4x + 3y = 11 \\ 2x - \frac{y}{3} = 0 \end{cases}$ for x and y .

Section 1.1 – Functions and Function Notation

Definition. A *function* is a rule that takes certain values as inputs and assigns to each input value exactly one output value.

Example. Let $y = \frac{2}{1+x}$.

x	0	1	2	3
y				

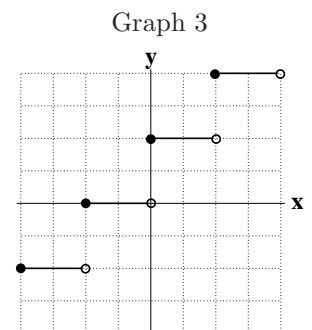
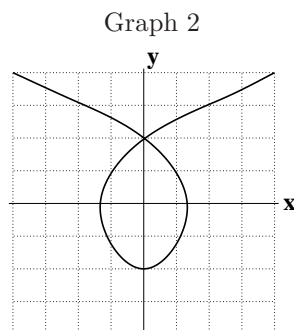
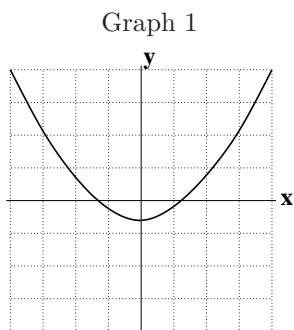
Example.

- t = time (in years) after the year 2000
- w = number of San Francisco '49er victories

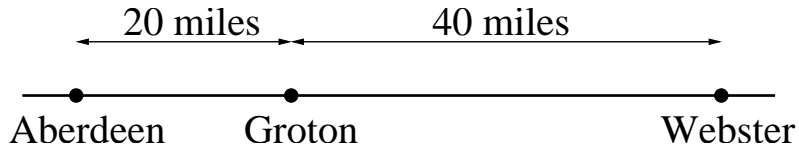
t	0	1	2	3	4	5	6	7	8	9
w	6	12	10	7	2	4	7	5	7	8

Observations:

Example. Which of the graphs below represent y as a function of x ?



Example. A woman drives from Aberdeen to Webster, South Dakota, going through Groton on the way, traveling at a constant speed for the whole trip. (See map below).



a. Sketch a graph of the woman's distance from Webster as a function of time.

b. Sketch a graph of the woman's distance from Groton as a function of time.

Section 1.2 – Rates of Change

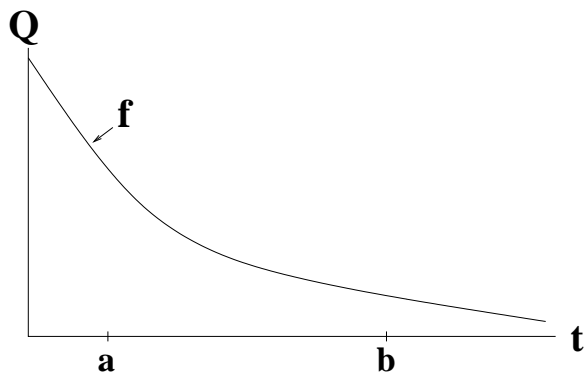
Preliminary Example. The table to the right shows the temperature, T , in Tucson, Arizona t hours after midnight.

t (hours after midnight)	0	3	4
T (temp. in °F)	85	76	70

Question. When does the temperature decrease the fastest: between midnight and 3 a.m. or between 3 a.m. and 4 a.m.?

Graphical Interpretation of Rate of Change

Definition. The average rate of change, or just rate of change of Q with respect to t is given by

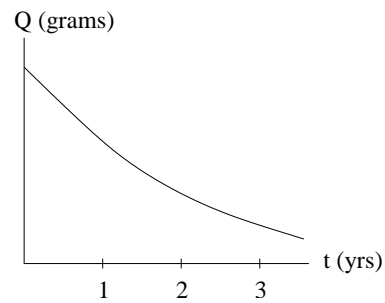


Alternate Formula for Rate of Change: The average rate of change of a function $Q = f(t)$ on the interval $a \leq t \leq b$ is given by the following formula:

Examples and Exercises

1. Let $f(x) = 4 - x^2$. Find the average rate of change of $f(x)$ on each of the following intervals.
- (a) $0 \leq x \leq 2$ (b) $2 \leq x \leq 4$ (c) $b \leq x \leq 2b$

2. To the right, you are given a graph of the amount, Q , of a radioactive substance remaining after t years. Only the t -axis has been labeled. Use the graph to give a **practical interpretation** of each of the three quantities that follow. A practical interpretation is an explanation of meaning using everyday language.

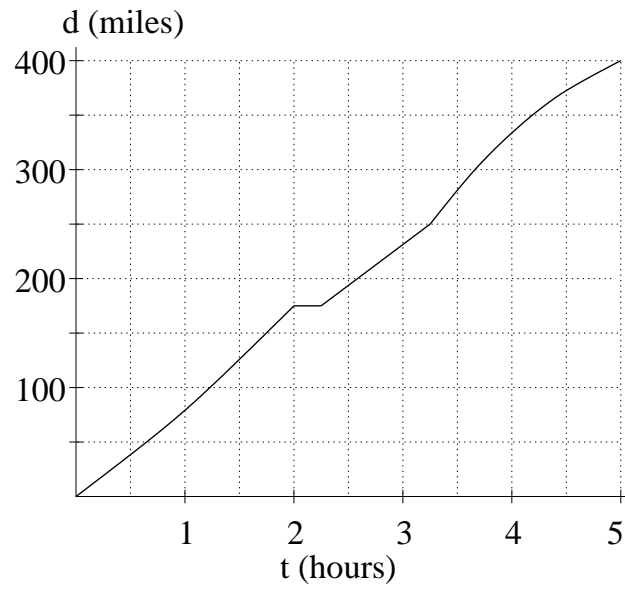


a. $f(1)$

b. $f(3)$

c. $\frac{f(3) - f(1)}{3 - 1}$

3. Two cars travel for 5 hours along Interstate 5. A South Dakotan in a 1983 Chevy Caprice travels 300 miles, always at a constant speed. A Californian in a 2009 Porsche travels 400 miles, but at varying speeds (see graph to the right).



- (a) On the axes above, sketch a graph of the distance traveled by the South Dakotan as a function of time.
- (b) Compute the average velocity of each car over the 5-hour trip.
- (c) Does the Californian drive faster than the South Dakotan over the entire 5 hour interval? Justify your answer!

Sections 1.3 & 1.4 – Linear Functions and Their Formulas

Preliminary Example. The cost, C , of your monthly phone bill consists of a \$30 basic charge, plus \$0.10 for each minute of long distance calls.

(a) Complete the table below, and sketch a graph.

t	0	30	60	90	120	150
C						

(b) Compute the average rate of change of C over any time interval.

(c) Find a formula for C in terms of t .

(d) If your bill is \$135, how long did you talk long distance?

Notes on Linear Functions:

1. If $y = f(x)$ is a linear function, then $y = mx + b$, where

2. If $y = f(x)$ is linear, then _____ input values produce output values.

Different forms for equations of lines:

Example 2. Find the slope and the y -intercept for each of the following linear functions.

(a) $3x + 5y = 20$

(b) $\frac{x - y}{5} = 2$

Examples and Exercises

1. Let $C = 20 - 0.35t$, where C is the cost of a case of apples (in dollars) t days after they were picked.

(a) Complete the table below:

t (days)	0	5	10	15
C (dollars)				

(b) What was the initial cost of the case of apples?

(c) Find the average rate of change of C with respect to t . Explain in practical terms (i.e., in terms of cost and apples) what this average rate of change means.

2. In parts (a) and (b) below, two different linear functions are described. Find a formula for each linear function, and write it in slope intercept form.

(a) The line passing through the points $(1, 2)$ and $(-1, 5)$.

(b)

C	10	15	20	25
F	50	59	68	77

3. According to one economic model, the demand for gasoline is a linear function of price. If the price of gasoline is $p = \$3.10$ per gallon, the quantity demanded in a fixed period of time is $q = 65$ gallons. If the price is $\$3.50$ per gallon, the quantity of gasoline demanded is 45 gallons for that period.
- Find a formula for q (demand) in terms of p (price).
 - Explain the economic significance of the slope in the above formula. In other words, give a practical interpretation of the slope.
 - According to this model, at what price is the gas so expensive that there is no demand?
 - Explain the economic significance of the vertical intercept of your formula from part (a).
4. Look back at your answer to problem 2(b). You might recognize this answer as the formula for converting Celsius temperatures to Fahrenheit temperatures. Use your formula to answer the following questions.
- Find C as a function of F .
 - What Celsius temperature corresponds to 90°F ?
 - Is there a number at which the two temperature scales agree?

Section 1.5 – Geometric Properties of Linear Functions

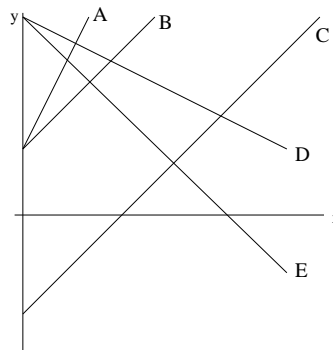
Example 1. You need to rent a car for one day and to compare the charges of 3 different companies. Company I charges \$20 per day with an additional charge of \$0.20 per mile. Company II charges \$30 per day with an additional charge of \$0.10 per mile. Company III charges \$70 per day with no additional mileage charge.

- (a) For each company, find a formula for the cost, C , of driving a car m miles in one day. Then, graph the cost functions for each company for $0 \leq m \leq 500$. (Before you graph, try to choose a range of C values would be appropriate.)

- (b) How many miles would you have to drive in order for Company II to be cheaper than Company I?

Example 2. Given below are the equations for five different lines. Match each formula with its graph to the right.

- $f(x) = 20 + 2x$
- $g(x) = 20 + 4x$
- $h(x) = 2x - 30$
- $u(x) = 60 - x$
- $v(x) = 60 - 2x$



Facts about the Line $y = mx + b$

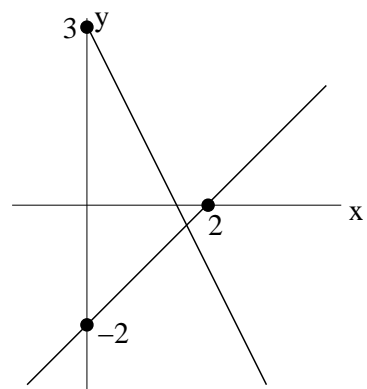
1. The y -intercept, b (also called the vertical intercept), tells us where the line crosses the _____.
2. If $m > 0$, the line _____ left to right. If $m < 0$, the line _____ left to right.
3. The larger the value of $|m|$ is, the _____ the graph.

Parallel and Perpendicular Lines

Fact: Two lines ($y = m_1x + b_1$ and $y = m_2x + b_2$) are ...

1. ... parallel if _____
2. ... perpendicular if _____

1. Consider the lines given in the figure to the right. Given that the slope of one of the lines is -2 , find the **exact coordinates** of the point of intersection of the two lines. (“Exact” means to leave your answers in fractional form.)



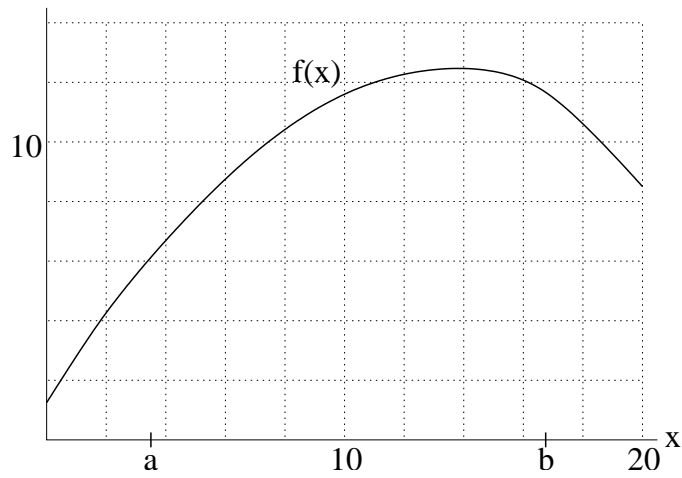
2. Parts (a) and (b) below each describe a linear function. Find a formula for the linear function described in each case.

(a) The line parallel to $2x - 3y = 2$ that goes through the point $(1, 1)$.

(b) The line perpendicular to $2x - 3y = 2$ that goes through the point $(1, 1)$.

Section 2.1 – Input and Output

Preliminary Example. Complete each of the following.



1. $f(10) =$ _____.
2. If $f(x) = 10$, then $x =$ _____.
3. $f(a) =$ _____.
4. $f(10) - f(6) =$ _____.

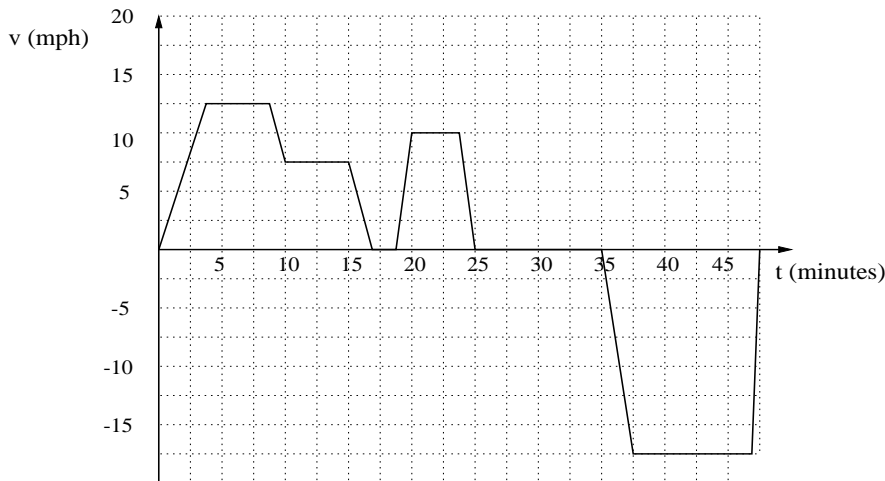
Examples and Exercises

1. The following table shows the amount of garbage produced in the U.S. as reported by the EPA.

t (years: 1960 \equiv 60)	60	65	70	75	80	85	90
G (millions of tons of garbage)	90	105	120	130	150	165	180

Consider the amount of garbage G as a function of time $G = f(t)$. Include units with your answers.

- $f(60) =$
 - $f(75) =$
 - Solve $f(t) = 165$.
2. Given is the graph of the function $v(t)$. It represents the velocity of a man riding his bike to the library and going back home after a little while. A negative velocity indicates that he is riding toward his house, away from the library.



Evaluate and interpret:

(a) $v(5) =$

Solve for t and interpret:

(d) $v(t) = 5$

(b) $v(40) =$

(e) $v(t) = -10$

(c) $v(12) - v(7) =$

(f) $v(t) = v(10)$

3. Consider the functions given below.

(a) Let $f(x) = x^2 - 2x - 8$.

i. Find $f(0)$.

ii. Solve $f(x) = 0$.

(b) Let $f(x) = \frac{1}{x+2} - 1$

i. Find $f(0)$.

ii. Solve $f(x) = 0$.

4. Let $f(x) = \frac{x}{x+1}$. Calculate and simplify $f\left(\frac{1}{t+1}\right)$, writing your final answer as a single fraction.

Section 2.2 – Domain and Range

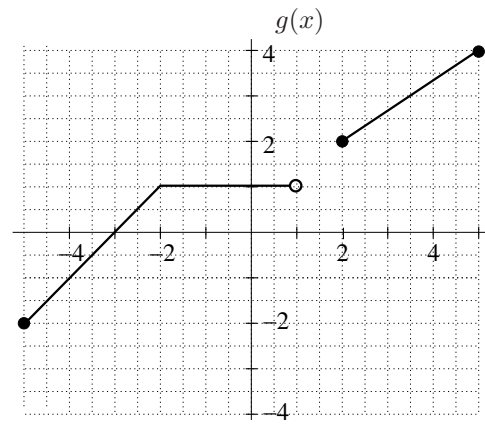
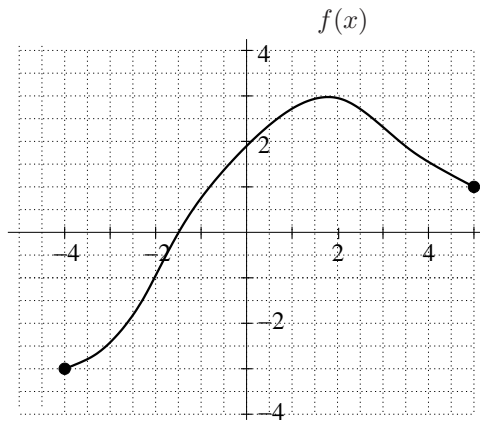
Definition. If $Q = f(t)$, then

1. The _____ of f is the set of all input values, t , that yield a meaningful output value.
2. The _____ of f is the corresponding set of all output values.

Example 1. Let $A = f(r)$ be the area, in cm^2 , of a circle of radius r cm. Find the domain and the range of f .

Example 2. Find the domain and range of the function $f(x) = \sqrt{x+2}$.

1. For each of the following functions below, give the domain and the range.



2. Oakland Coliseum is capable of seating 63,026 fans. For each game, the amount of money that the Raider's organization makes is a function of the number of people, n , in attendance. If each ticket costs \$30.00, find the domain and range of this function. Sketch its graph.

3. Find the domain and range of each of the following functions.

(a) $f(x) = \sqrt{3x+7}$

(c) $h(x) = x^2 - x - 6$

(b) $g(x) = \frac{1}{(x-1)^2}$

(d) $k(x) = \sqrt{x^2 - x - 6}$

Section 2.4 – Inverse Functions

Preliminary Example. Recall the phone example from earlier, where a calling plan charged us a \$30 monthly service fee and then \$0.10 per minute for long distance calls.

t	0	30	33	36	60
C	30	33	33.30	33.60	36

For each of the following, fill in the blank and then give an interpretation of the entire statement.

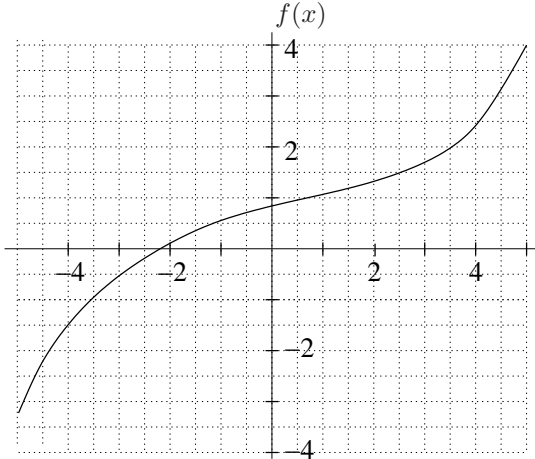
(a) $f(36) = \underline{\hspace{2cm}}$

(b) $f^{-1}(36) = \underline{\hspace{2cm}}$

(c) $f^{-1}(\underline{\hspace{2cm}}) = 33$

Examples and Exercises

1. Use the two functions shown below to fill in the blanks to the right.



- (a) $f(2) = \underline{\hspace{2cm}}$ (b) $f^{-1}(2) = \underline{\hspace{2cm}}$
 (c) $g(0) = \underline{\hspace{2cm}}$ (d) $g^{-1}(0) = \underline{\hspace{2cm}}$
 (e) $f(3) + 1 = \underline{\hspace{2cm}}$ (f) $f^{-1}(3) + 1 = \underline{\hspace{2cm}}$
 (g) $f(3 + 1) = \underline{\hspace{2cm}}$ (h) $f^{-1}(3 + 1) = \underline{\hspace{2cm}}$
 (i) If $g^{-1}(x) = 0$, then $x = \underline{\hspace{2cm}}$.

x	-6	-4	-2	0	2	4	6
$g(x)$	2	0	3	7	6	1	5

2. Let $A = f(n)$ be the amount of periwinkle blue paint, in gallons, needed to paint n square feet of a house. Explain in practical terms what each of the following quantities represents. Use a complete sentence in each case.

(a) $f(20)$

(b) $f^{-1}(20)$

3. If a cricket chirps R times per minute, then the outside temperature is given by $T = f(R) = \frac{1}{4}R + 40$ degrees Fahrenheit.

(a) Find a formula for the inverse function $R = f^{-1}(T)$.

(b) Calculate and interpret $f(50)$ and $f^{-1}(50)$.

Section 2.5 – Concavity

Definitions.

1. A function $f(x)$ is called *increasing* if its graph _____ from left to right. It is called *decreasing* if its graph _____ from left to right.
2. A function $f(x)$ is called *concave up* if its average rate of change increases from left to right.
3. A function $f(x)$ is called *concave down* if its average rate of change decreases from left to right.

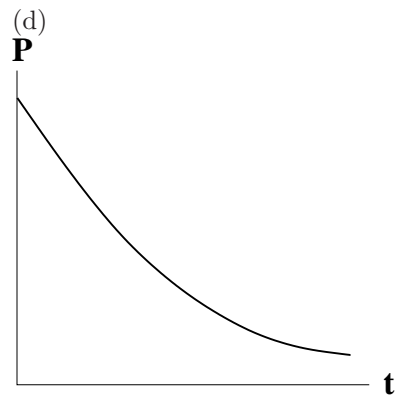
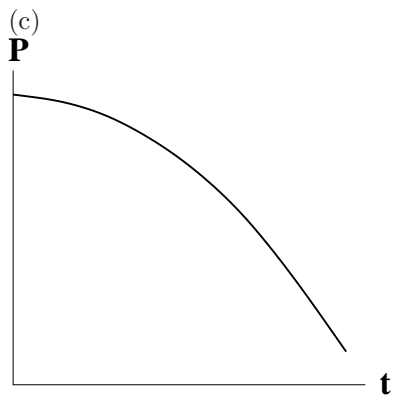
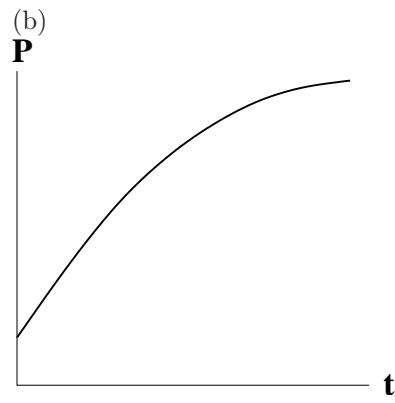
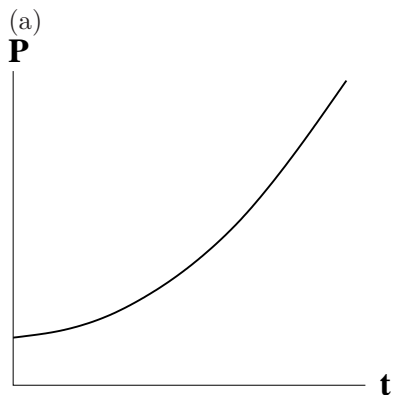
Describe the shape of the graph of a function $f(x)$ that is concave up:

Describe the shape of the graph of a function $f(x)$ that is concave down:

Example. Read the following description of a function. Then, decide whether the function is increasing or decreasing. What does the scenario tell you about the concavity of the graph modeling it?

“When a new product is introduced, the number of people who use the product increases slowly at first, and then the rate of increase is faster (as more and more people learn about the product). Eventually, the rate of increase slows down again (when most people who are interested in the product are already using it).”

Example. Consider the following graphs of population, P , as a function of time, t .



Descriptions

(a) P is _____, and the rate of change of P is _____.

(b) P is _____, and the rate of change of P is _____.

(c) P is _____, and the rate of change of P is _____.

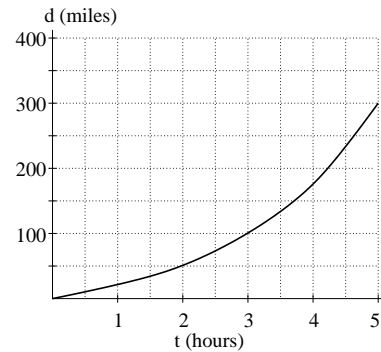
(d) P is _____, and the rate of change of P is _____.

Examples and Exercises

1. Consider the functions shown below. Fill in the accompanying tables and then decide whether each function is increasing or decreasing, and whether it is concave up or concave down.

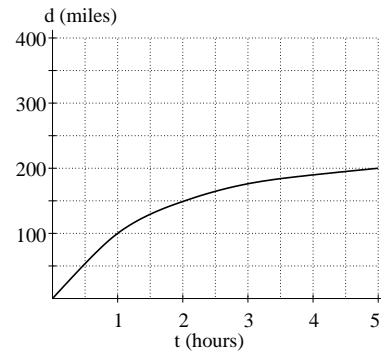
- (a) **Description.** This graph gives distance driven as a function of time for a California driver.

t	0	2	3	5
d				
$\frac{\Delta d}{\Delta t}$				



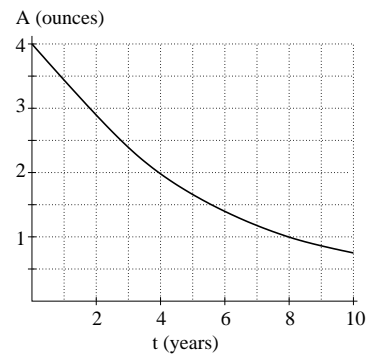
- (b) **Description.** This graph gives distance driven as a function of time for a South Dakota driver.

t	0	2	3	5
d				
$\frac{\Delta d}{\Delta t}$				



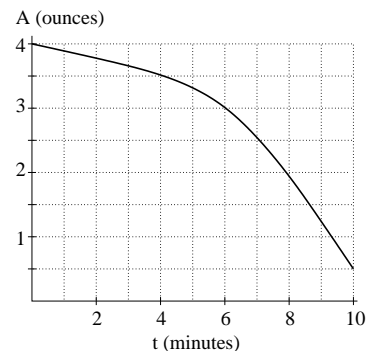
- (c) **Description.** This graph gives the amount of a decaying twinkie as a function of time.

t	0	4	6	10
A				
$\frac{\Delta A}{\Delta t}$				



- (d) **Description.** This graph gives the amount of ice remaining in a melting ice cube as a function of time.

t	0	4	6	10
A				
$\frac{\Delta A}{\Delta t}$				

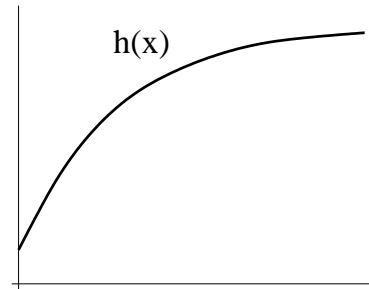


2. Decide whether each of the following functions are concave up, concave down, or neither.

x	0	1	2	3	4
$f(x)$	1	3	6	10	20

x	0	1	2	3	4
$g(x)$	10	9	7	4	0

$$p(x) = 3x + 1$$



Section 2.6 – Quadratic Functions

Definition. The general form of a *quadratic function* is given by

Some quadratic functions can also be written in the factored form

Note: The graph of a quadratic function is a _____.

Example. Find the zeros (if possible) of the following quadratic functions.

(a) $f(x) = 2x^2 + x - 3$

(b) $g(x) = x^2 + 4x + 2$

Examples and Exercises

1. Find (if possible), the zeros of the following quadratic functions.

(a) $f(x) = x^2 + 5x - 14$

(b) $g(x) = x^2 + 1$

2. The height of a rock thrown into the air is given by $h(t) = 40t - 16t^2$ feet, where t is measured in seconds.

(a) Calculate $h(1)$ and give a practical interpretation of your answer.

(b) Calculate the zeros of $h(t)$ and explain their meaning in the context of this problem.

(c) Solve the equation $h(t) = 10$ and explain the meaning of your solutions in the context of this problem.

(d) Use a graph of $h(t)$ to estimate the maximum height reached by the stone. When, approximately, does the stone reach its maximum height? Is the function concave up or concave down?

Chapter 3 Tools – Exponents

Properties of Exponents

1. $a^n a^m =$ _____

3. $(a^m)^n =$ _____

5. $\left(\frac{a}{b}\right)^n =$ _____

2. $\frac{a^n}{a^m} =$ _____

4. $(ab)^n =$ _____

Caution!!

Some Definitions

(A) $a^0 =$ _____

(B) $a^{-n} =$ _____

(C) $a^{\frac{1}{n}} =$ _____

(D) $a^{\frac{m}{n}} =$ _____

Example 1. Without a calculator, simplify $9^{-1/2} + \sqrt{0.01}$.

Example 2. Simplify $\sqrt{x^e y^{e/2}} + (x^e)(x^e)^2$.

Example 3. Simplify both of the following: (a) $\frac{n^{-1}a}{a^2}$ (b) $\frac{n^{-1}a + 1}{a^2}$

Examples and Exercises

Directions. For problems 1-7, simplify. For problem 8, solve for x . You may need extra paper for your calculations.

1. $\frac{(xy^3)^2}{x^0y^5}$

5. $\frac{2M + M^{-1}}{1 + 2M^{-2}}$

2. $\frac{(AB)^4}{A^{-1}B^{-2}}$

6. $3\sqrt[3]{t^3 + 7(t^9)^{1/3}}$

3. $\frac{a^3b^{-1}}{\sqrt{a^{5/2}}}$

7. $\frac{2km^3 + k^2m}{km^{-1}}$

4. $2b^{-1}(b^2 + b) - 2$

8. $81^x = 3$

Sections 3.1-3.3 – Exponential Functions

Example 1. The population of a rapidly-growing country starts at 5 million and increases by 10% each year. Complete the table below:

t (years)	P , population (in millions)	ΔP , increase in population (mil)
0		
1		
2		
3		
4		

Definition. An exponential function $Q = f(t)$ has the formula $f(t) = ab^t$, $b > 0$,

where

$$a =$$

$$b =$$

Note: $b = 1 + r$, where r is the decimal representation of the percent rate of change.

Example 2.

Description	Growth Factor and Formula
The population, P , of ants in your kitchen starts at 10 and increases by 5% per day.	
The value, V , of a 1982 Chevy Caprice starts at \$10000 and decreases by 8% per year.	
The air pressure, A , starts at _____ millibars at sea level ($h = 0$) and decreases by _____ per mile increase in elevation.	$A = 960(0.8)^h$

Example 3. Analyze the functions f and g below. Which is linear? Which is exponential? Give a formula for each function.

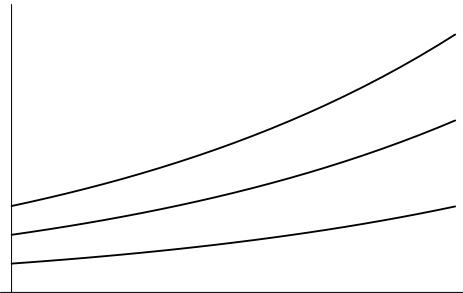
x	5	10	15	20	25
$f(x)$	10	17	24	31	38
$g(x)$	100	115	132.25	152.09	174.9

Comparison of Linear and Exponential Functions. If $y = f(x)$ is given as a table of values, and if the x -values are equally spaced, then

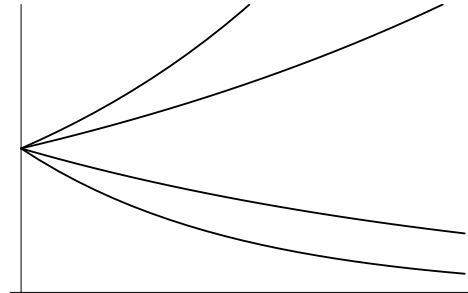
1. f is linear if the _____ of successive y -values is constant.
2. f is exponential if the _____ of successive y -values is constant.

Example 4.

Below are the graphs of $Q = 150(1.2)^t$, $Q = 50(1.2)^t$, and $Q = 100(1.2)^t$. Match each formula to the correct graph.



Below are the graphs of $Q = 50(1.2)^t$, $Q = 50(0.6)^t$, $Q = 50(0.8)^t$, and $Q = 50(1.4)^t$. Match each formula to the correct graph.



Observations about the graph of $Q = ab^t$:

Examples and Exercises

1. Suppose we start with 100 grams of a radioactive substance that decays by 20% per year. First, complete the table below. Then, find a formula for the amount of the substance as a function of t and sketch a graph of the function.

t (years)	0	1	2	3	4
Q (grams)					

2. Suppose you invest \$10000 in the year 2000 and that the investment earns 4.5% interest annually.

(a) Find a formula for the value of your investment, V , as a function of time.

(b) What will the investment be worth in 2010? in 2020? in 2030?

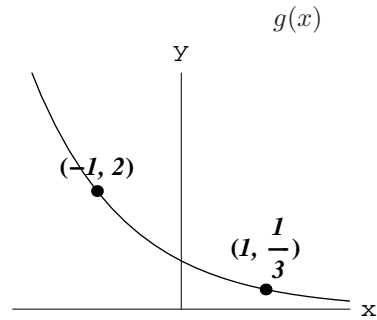
3. The populations of the planet Vulcan and the planet Romulus are recorded in 1980 and in 1990 according to the table below. Also, assume that the population of Vulcan is growing exponentially and that the population of Romulus is growing linearly.

Planet	1980 Population (billions)	1990 Population (billions)
Vulcan	8	12
Romulus	16	20

- (a) Find two formulas; one for the population of Vulcan as a function of time and one for the population of Romulus as a function of time. Let $t = 0$ denote the year 1980.
- (b) Use your formulas to predict the population of both planets in the year 2000.
- (c) According to your formula, in what year will the population of Vulcan reach 50 billion? Explain how you got your answer.
- (d) In what year does the population of Vulcan overtake the population of Romulus? Justify your answer with an accurate graph and an explanation.

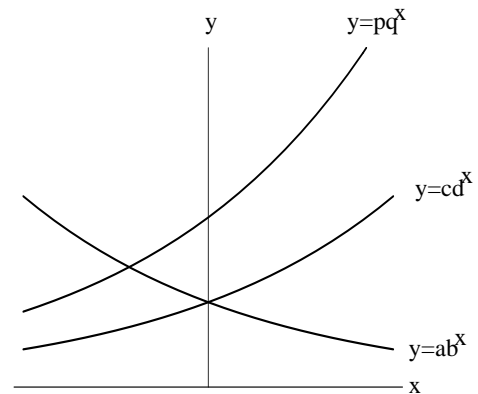
4. Find possible formulas for each of the two functions f and g described below.

x	0	2	4	6
$f(x)$	2	2.5	3.125	3.90625



5. Consider the exponential graphs pictured below and the six constants $a, b, c, d, p,$ and q .

(a) Which of these constants are **definitely** positive?



(b) Which of these constants are **definitely** between 0 and 1?

(c) Which two of these constants are **definitely** equal?

(d) Which one of the following pairs of constants **could be** equal?

a and p b and d b and q d and q

Section 3.4 – Continuous Growth and the Number e

Preliminary Example. At the In-Your-Dreams Bank of America, all investments earn 100% interest annually. Suppose that you invest \$1000 at a time that we will call month 0. Fill in the blanks below to compare what your investment will be worth 1 year later using various methods of interest compounding.

Month	Compounded 1 Time	Compounded 2 Times	Compounded 4 Times
0	\$1000	\$1000	\$1000
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

Alternative Formula for Exponential Functions. Given an exponential function $Q = ab^t$, it is possible to rewrite Q as follows:

$$Q = \underline{\hspace{10em}}$$

The constant k is then called the *continuous growth rate* of Q .

Notes:

- If $k > 0$, then Q is increasing.
- If $k < 0$, then Q is decreasing.

Exercise Suppose that the population of a town starts at 5000 and grows at a continuous rate of 2% per year.

(a) Write a formula for the population of the town as a function of time, in years, after the starting point.

(b) What will the population of the town be after 10 years?

(c) By what percentage does the population of the town grow each year?

Section 4.1 – Logarithms and Their Properties

Definition. If x is a positive number, then

1. $\log x$ is

2. $\ln x$ is

In other words,

$y = \log x$ means that

$y = \ln x$ means that

Example 1. Calculate the following, exactly if possible.

(a) $\log 100$

(c) $\ln e$

(e) $\log 5$

(b) $\log 0.1$

(d) $\ln(e^2)$

Facts about Logarithms.

1. (a) $\log(10^x) =$

$\ln(e^x) =$

(b) $10^{\log x} =$

$e^{\ln x} =$

2. (a) $\log(ab) =$

$\ln(ab) =$

(b) $\log\left(\frac{a}{b}\right) =$

$\ln\left(\frac{a}{b}\right) =$

(c) $\log(b^t) =$

$\ln(b^t) =$

3. (a) $\log 1 =$

$\ln 1 =$

(b) $\log 10 =$

$\ln e =$

Example 2. Solve each of the following equations for x .

(a) $5 \cdot 4^x = 25$

(b) $5x^4 = 25$

Examples and Exercises

1. Solve each of the following equations for x .

(a) $5 \cdot 3^x = 2 \cdot 7^x$

(d) $5x^9 = 10$

(b) $10e^{4x+1} = 20$

(e) $e^{2x} + e^{2x} = 1$

(c) $a \cdot b^t = c \cdot d^{2t}$

(f) $\ln(x + 5) = 10$

2. Simplify each of the following expressions.

(a) $\log(2A) + \log(B) - \log(AB)$

(b) $\ln(ab^t) - \ln((ab)^t) - \ln a$

3. Decide whether each of the following statements are true or false.

(a) $\ln(x + y) = \ln x + \ln y$

(b) $\ln(x + y) = (\ln x)(\ln y)$

(c) $\ln(ab^2) = \ln a + 2 \ln b$

(d) $\ln(ab^x) = \ln a + x \ln b$

(e) $\ln(1/a) = -\ln a$

Section 4.2 – Logarithms and Exponential Models

Review: Two ways of writing exponential functions:

$$(1) Q = ab^t$$

$$(2) Q = ae^{kt}$$

Example 1. Fill in the gaps in the chart below, assuming that t is measured in years:

Formula		Growth or Decay Rate	
$Q = ab^t$	$Q = ae^{kt}$	Per Year	Continuous Per Year
	$Q = 6e^{-0.04t}$		
$Q = 5(1.2)^t$			
$Q = 10(0.91)^t$			

Example 2. The population of a bacteria colony starts at 100 and grows by 30% per hour.

(a) Find a formula for the number of bacteria, P , after t hours.

(b) What is the doubling time for this population; that is, how long does it take the population to double in size?

(c) What is the continuous growth rate of the colony?

Definition. The half-life of a radioactive substance is the amount of time that it takes for

Example 3. The half-life of a Twinkie is 14 days.

(a) Find a formula for the amount of Twinkie left after t days.

(b) Find the daily decay rate of the Twinkie.

Examples and Exercises

1. (Taken from Connally) Scientists observing owl and hawk populations collect the following data. Their initial count for the owl population is 245 owls, and the population grows by 3% per year. They initially observe 63 hawks, and this population doubles every 10 years.

(a) Find formulas for the size of the population of owls and hawks as functions of time.

(b) When will the populations be equal?

2. Find the half-lives of each of the following substances.

(a) Tritium, which decays at an annual rate of 5.471% per year.

(b) Vikinium, which decays at a continuous rate of 10% per week.

3. If 17% of a radioactive substance decays in 5 hours, how long will it take until only 10% of a given sample of the substance remains?

Section 4.3 – The Logarithmic Function

1. Consider the functions $f(x) = \ln x$ and $g(x) = \log x$.

(a) Complete the table below.

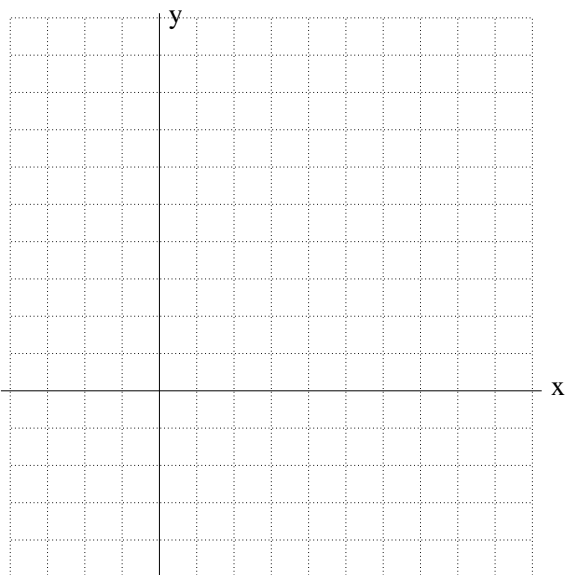
x	0.1	0.5	1	2	4	6	8	10
$\ln x$								
$\log x$								

(b) Plug a few very small numbers x into $\ln x$ and $\log x$ (like 0.01, 0.001, etc.) What happens to the output values of each function?

(c) If you plug in $x = 0$ or negative numbers for x , are $\ln x$ and $\log x$ defined? Explain.

(d) What is the domain of $f(x) = \ln x$? What is the domain of $g(x) = \log x$?

(e) Sketch a graph of $f(x) = \ln x$ below, choosing a reasonable scale on the x and y axes. Does $f(x)$ have any vertical asymptotes? Any horizontal asymptotes?



2. What is the domain of the following four functions?

(a) $y = \ln(x^2)$

(b) $y = (\ln x)^2$

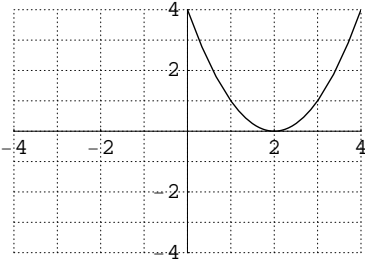
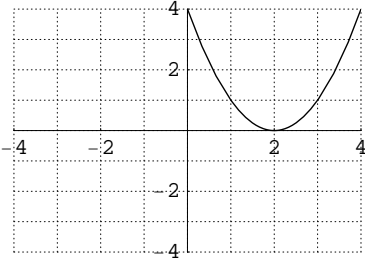
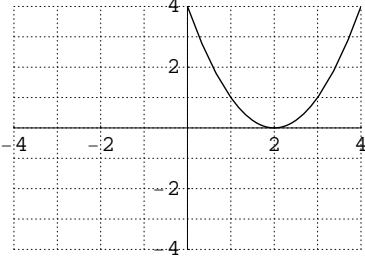
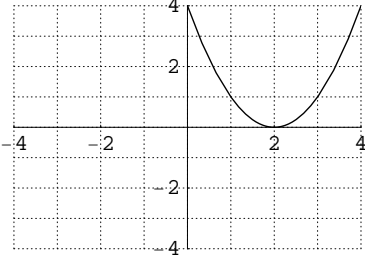
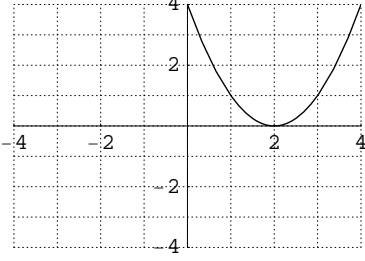
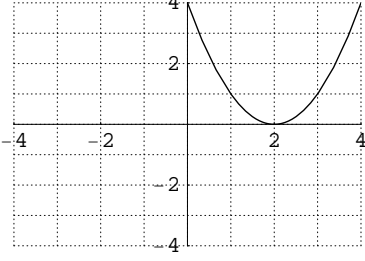
(c) $y = \ln(\ln x)$

(d) $y = \ln(x - 3)$

3. Consider the exponential functions $f(x) = e^x$ and $g(x) = e^{-x}$. What are the domains of these two functions? Do they have any horizontal asymptotes? any vertical asymptotes?

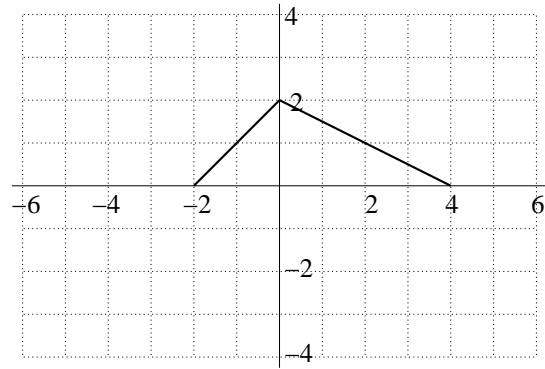
Sections 5.1-5.3 – Function Transformations

Example 1. Consider the function $f(x) = x^2 - 4x + 4$.

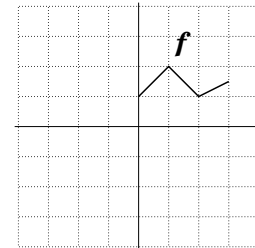
Transformation	Formula	Graph	Description
$y = f(x) + 2$			
$y = f(x) - 2$			
$y = f(x + 2)$			
$y = f(x - 2)$			
$y = f(-x)$			
$y = -f(x)$			

Example 2. Let $y = f(x)$ be the function whose graph is given to the right. Sketch the graphs of the transformations $y = f(x-2)$, $y = -2f(x)$, and $y = f(-x)$. Then, fill in the entries in the table below.

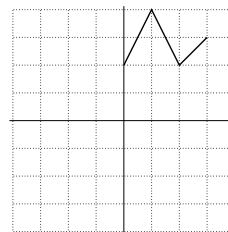
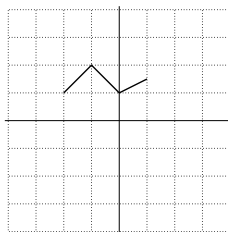
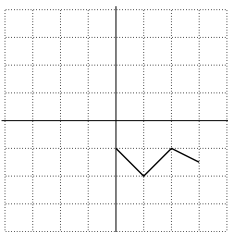
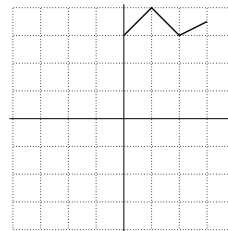
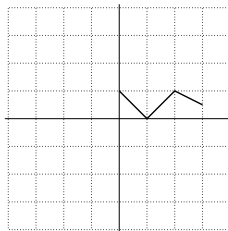
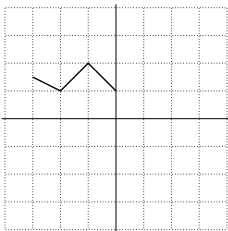
x	-4	-2	0	2	4	6
$f(x)$						
$f(x-2)$						
$-2f(x)$						
$f(-x)$						



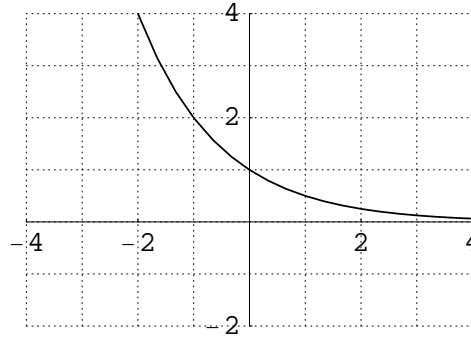
Example 3. To the right, you are given the graph of a function f . Match each graph below to the appropriate transformation formula. Note that some transformation formulas will not match any of the graphs.



- (a) $y = 2f(x)$ (d) $y = 2f(x) + 2$ (g) $y = -f(x)$
 (b) $y = f(x) + 2$ (e) $y = f(x + 2)$ (h) $y = 4f(x)$
 (c) $y = 2 - f(x)$ (f) $y = f(-x)$

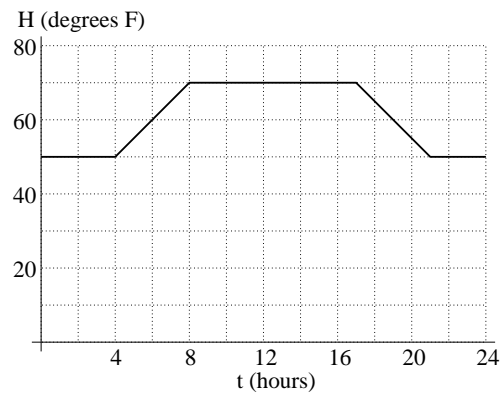


3. Given to the right is the graph of the function $y = \left(\frac{1}{2}\right)^x$. On the same set of axes, sketch the graph of $y = \left(\frac{1}{2}\right)^{x-2}$ and $y = \left(\frac{1}{2}\right)^x - 2$.



4. Let $H = f(t)$ be the temperature of a heated office building t hours after midnight. (See diagram to the right for a graph of f .) Write down a formula for a new function that matches each story below.

- (a) The manager decides that the temperature should be lowered by 5 degrees throughout the day.



- (b) The manager decides that employees should come to work 2 hours later and leave 2 hours later.

Definition

We say that a function is *even* if $f(-x) = f(x)$ for all x in the domain of the function. In other words, an even function is symmetric about the

_____.

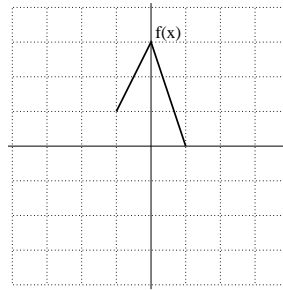
Sketch

We say that a function is *odd* if $f(-x) = -f(x)$ for all x in the domain of the function. In other words, an odd function is symmetric about the

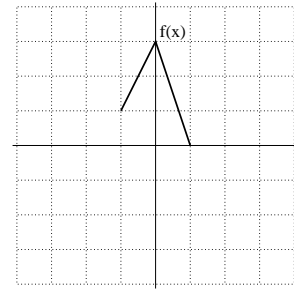
_____.

5. Use algebra to show that $f(x) = x^4 - 2x^2 + 1$ is an even function and that $g(x) = x^3 - 5x$ is an odd function.

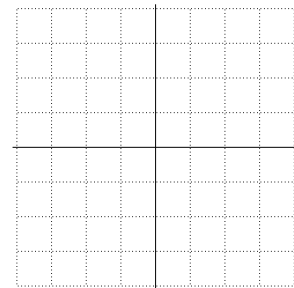
6. Given the graph of $y = f(x)$ given to the right, sketch the graph of the following related functions:



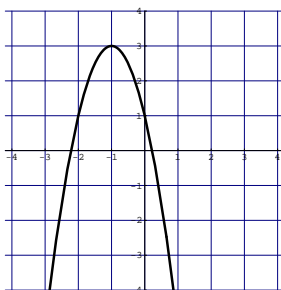
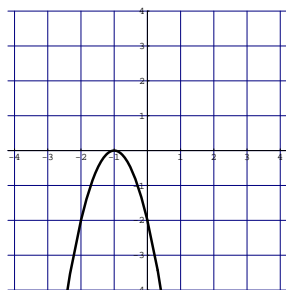
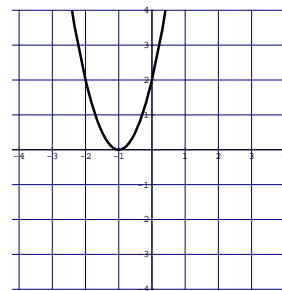
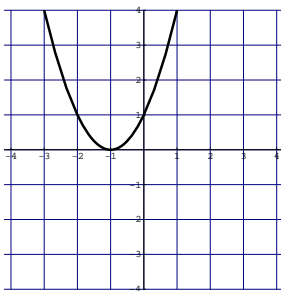
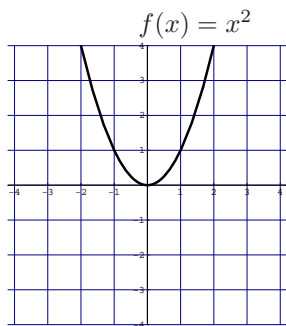
(a) $y = -f(x + 3) + 1$



(b) $y = 2 - f(1 - x)$



Section 5.5 – The Family of Quadratic Functions



Information about Quadratic Functions

In general, a *quadratic function* f can be written in several different ways:

1. $f(x) = ax^2 + bx + c$ (standard form, where a, b , and c are constant)
2. $f(x) = a(x - r)(x - s)$ (factored form, where a, r , and s are constant)
3. $f(x) = a(x - h)^2 + k$ (vertex form, where a, h , and k are constant)

Notes.

- The graph of a quadratic function is called a _____.
- In factored form, the numbers r and s represent the _____ of f .
- In vertex form, the point (h, k) is called the _____ of the parabola. The axis of symmetry is the line _____ . The graph opens upward if _____ and downward if _____.

Example 1. Find the vertex of $y = x^2 - 8x - 27$ by completing the square.

Example 2. Find the vertex of $y = 2x^2 + 7x + 3$ by completing the square.

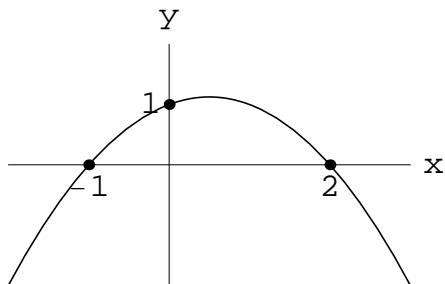
Examples and Exercises

1. For each of the following, complete the square in order to find the vertex.

(a) $y = x^2 - 40x + 1$

(b) $y = 2x^2 + 12x + 3$

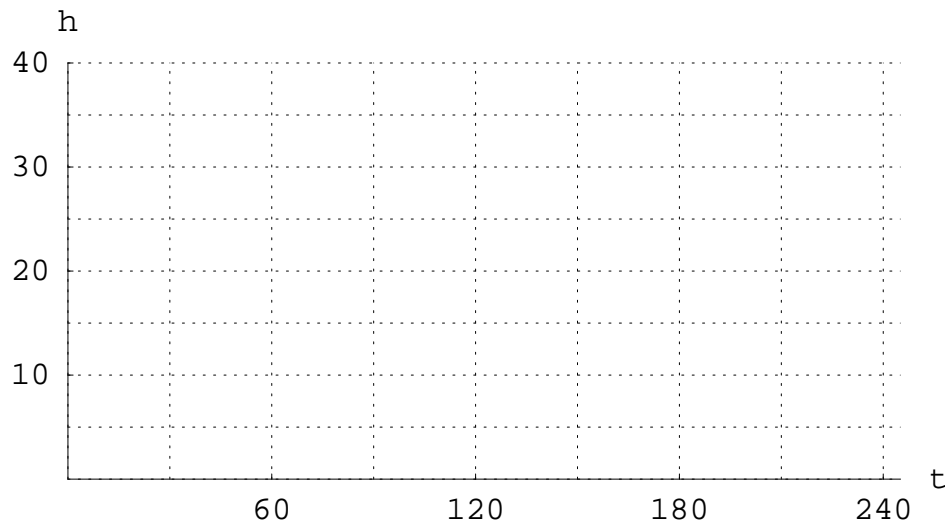
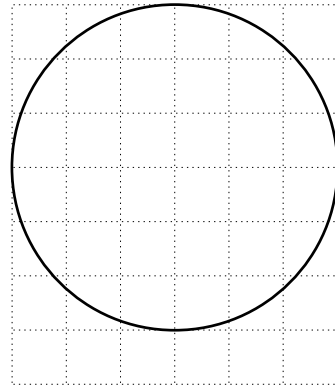
2. Find a formula for the quadratic function shown below. Also find the vertex of the function.



3. A parabola has its vertex at the point $(2, 3)$ and goes through the point $(6, 11)$. Find a formula for the parabola.
4. (Taken from Connally) A tomato is thrown vertically into the air at time $t = 0$. Its height, $d(t)$ (in feet), above the ground at time t (in seconds) is given by $d(t) = -16t^2 + 48t$.
- (a) Find t when $d(t) = 0$. What is happening to the tomato the first time that $d(t) = 0$? The second time?
- (b) When does the tomato reach its maximum height? How high is the tomato's maximum height?

Section 6.1 – Periodic Functions

Preliminary Example. The Brown County Ferris Wheel has diameter 30 meters and completes one full revolution every two minutes. When you are at the lowest point on the wheel, you are still 5 meters above the ground. Assuming you board the ride at $t = 0$ seconds, sketch a graph of your height, $h = f(t)$, as a function of time.



Definition. A function f is called *periodic* if its output values repeat at regular intervals. Graphically, this means that if the graph of f is shifted horizontally by p units, the new graph is identical to the original. Given a periodic function f :

1. The *period* is the horizontal distance that it takes for the graph to complete one full cycle. That is, if p is the period, then $f(t + p) = f(t)$.
2. The *midline* is the horizontal line midway between the function's maximum and minimum output values.
3. The *amplitude* is the vertical distance between the function's maximum value and the midline.

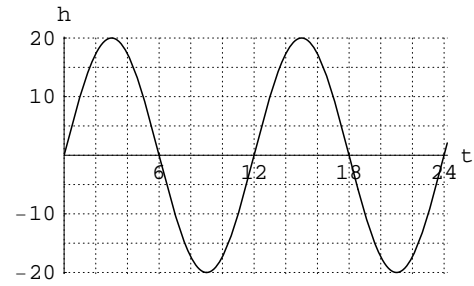
Example 1. What are the amplitude, midline and period of the function $h = f(t)$ from the preliminary example?

Examples and Exercises

1. The function given below models the height, h , in feet, of the tide above (or below) mean sea level t hours after midnight.

(a) Is the tide rising or falling at 1:00 a.m.?

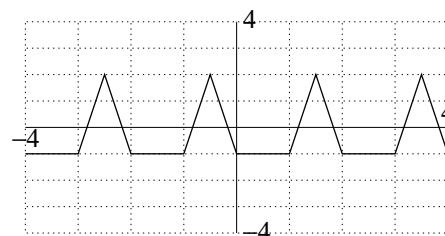
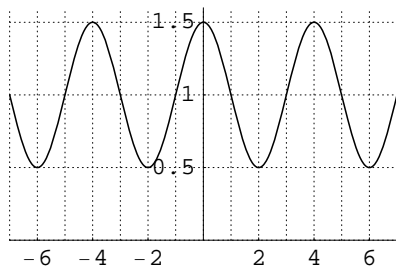
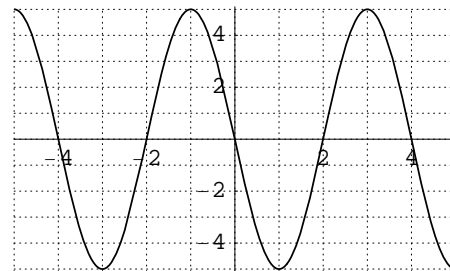
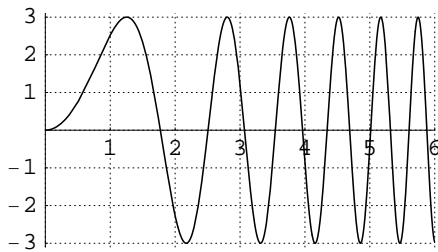
(b) When does low tide occur?



(c) What is the amplitude of the function? Give a practical interpretation of your answer.

(d) What is the midline of the function? Give a practical interpretation of your answer.

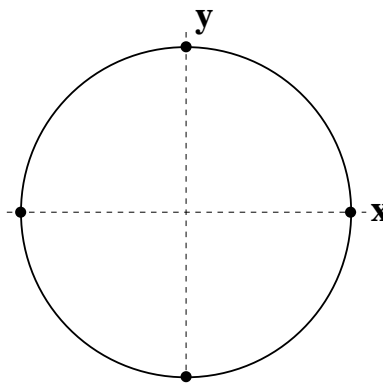
2. Which of the following functions are periodic? For those that are, what is the period?



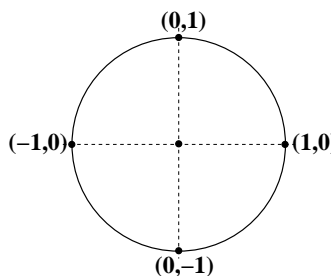
Section 6.2 – The Sine and Cosine Functions

Angle Measurement in Circles

- Angles start from the positive x -axis.
- Counterclockwise defined to be positive.

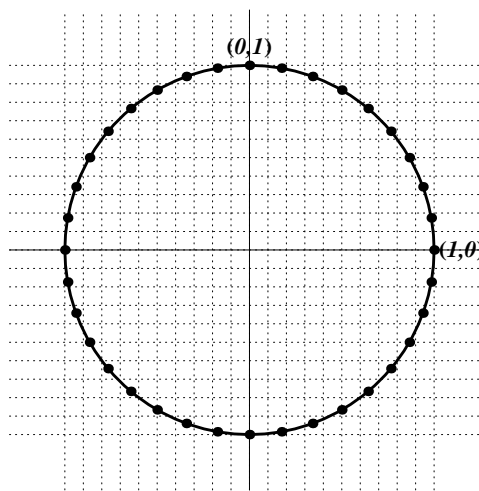


Definition. The *unit circle* is the term used to describe a circle that has its center at the origin and has radius equal to 1. The *cosine* and *sine* functions are then defined as described below.



Example 1. On the unit circle to the right, the angles 10° , 20° , 30° , etc., are indicated by black dots on the circle. Use this diagram to estimate each of the following:

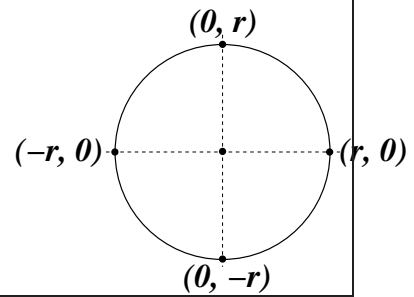
- (a) $\cos(30^\circ) = \underline{\hspace{2cm}}$
- (b) $\sin(150^\circ) = \underline{\hspace{2cm}}$
- (c) $\cos(270^\circ) = \underline{\hspace{2cm}}$



Example 2.

- (a) Find an angle θ between 0° and 360° that has the same sine as 40° .
- (b) Find an angle θ between 0° and 360° that has the same cosine as 40° .

Theorem. Consider a circle of radius r centered at the origin. Then the x and y coordinates of a point on this circle are given by the following formulas:



Examples and Exercises

1. Use the unit circle to the right to estimate each of the following quantities to the nearest 0.05 of a unit.

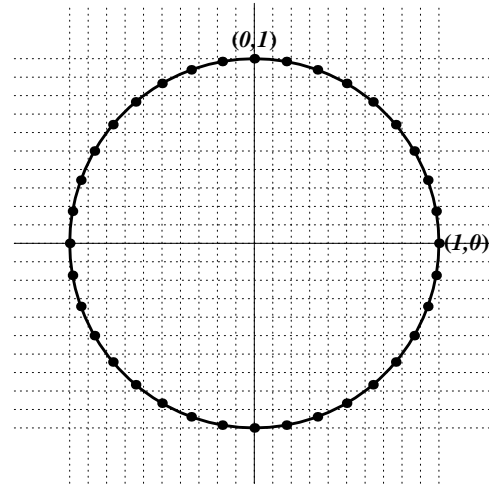
(a) $\sin(90^\circ) = \underline{\hspace{2cm}}$ (b) $\cos(90^\circ) = \underline{\hspace{2cm}}$

(c) $\sin(180^\circ) = \underline{\hspace{2cm}}$ (d) $\cos(180^\circ) = \underline{\hspace{2cm}}$

(e) $\cos(45^\circ) = \underline{\hspace{2cm}}$ (f) $\sin(-90^\circ) = \underline{\hspace{2cm}}$

(g) $\cos(70^\circ) = \underline{\hspace{2cm}}$ (h) $\sin(190^\circ) = \underline{\hspace{2cm}}$

(i) $\sin(110^\circ) = \underline{\hspace{2cm}}$ (j) $\cos(110^\circ) = \underline{\hspace{2cm}}$



2. For each of the following, fill in the blank with an angle between 0° and 360° , different from the first one, that makes the statement true.

(a) $\sin(20^\circ) = \sin(\underline{\hspace{2cm}})$ (b) $\sin(70^\circ) = \sin(\underline{\hspace{2cm}})$ (c) $\sin(225^\circ) = \sin(\underline{\hspace{2cm}})$

(d) $\cos(20^\circ) = \cos(\underline{\hspace{2cm}})$ (e) $\cos(70^\circ) = \cos(\underline{\hspace{2cm}})$ (f) $\cos(225^\circ) = \cos(\underline{\hspace{2cm}})$

3. Given to the right is a unit circle. Fill in the blanks with the correct answer in terms of a or b .

(a) $\sin(\theta + 360^\circ) = \underline{\hspace{2cm}}$

(b) $\sin(\theta + 180^\circ) = \underline{\hspace{2cm}}$

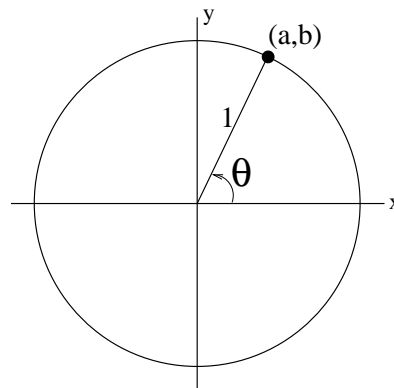
(c) $\cos(180^\circ - \theta) = \underline{\hspace{2cm}}$

(d) $\sin(180^\circ - \theta) = \underline{\hspace{2cm}}$

(e) $\cos(360^\circ - \theta) = \underline{\hspace{2cm}}$

(f) $\sin(360^\circ - \theta) = \underline{\hspace{2cm}}$

(g) $\sin(90^\circ - \theta) = \underline{\hspace{2cm}}$

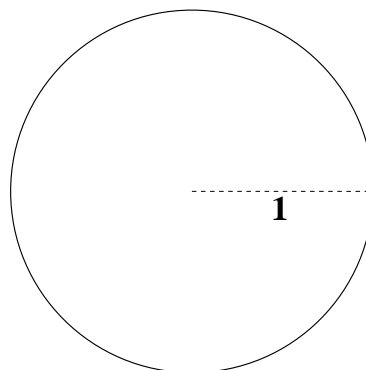


4. Use your calculator to find the coordinates of the point P at the given angle on a circle of radius 4 centered at the origin.

- (a) 70° (b) 255°

Section 6.3 – Radian Measure

Definition. An angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1.



Conversion Factors:

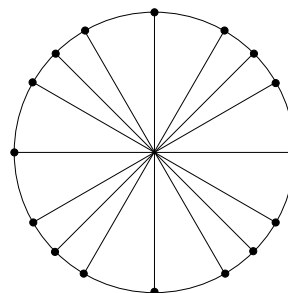
Degrees \times \longrightarrow Radians

Radians \times \longrightarrow Degrees

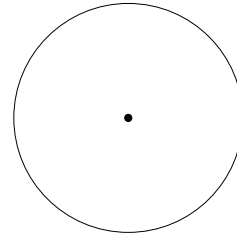
Example 2. Convert each of the following angles from radians to degrees or from degrees to radians. An angle measure is assumed to be in radians if the degree symbol is not indicated after it.

- (a) 30° (b) $\frac{3\pi}{2}$ (c) 1.4

Example 3. On the unit circle to the right, label the indicated “common” angles with their degree and radian measures.

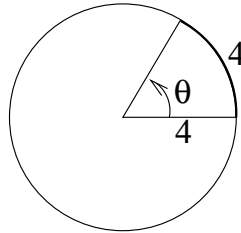
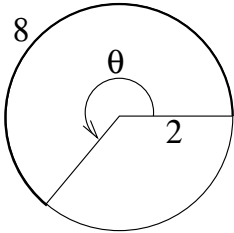


Theorem. The arc length, s , spanned in a circle of radius r by an angle of θ radians, $0 \leq \theta \leq 2\pi$, is given by

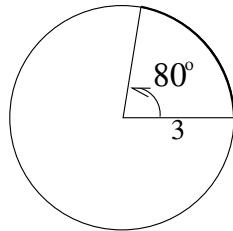
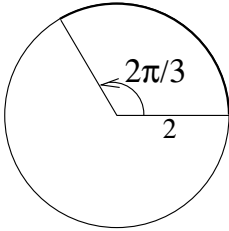


Examples and Exercises

1. In the pictures below, you are given the radius of a circle and the length of a circular arc cut off by an angle θ . Find the degree and radian measure of θ .



2. In the pictures below, find the length of the arc cut off by each angle.



3. A satellite orbiting the earth in a circular path stays at a constant altitude of 100 kilometers throughout its orbit. Given that the radius of the earth is 6370 kilometers, find the distance that the satellite travels in completing 70% of one complete orbit.

4. An ant starts at the point $(0,3)$ on a circle of radius 3 (centered at the origin) and walks 2 units counterclockwise along the arc of the circle. Find the x and the y coordinates of where the ant ends up.

Section 6.4 Supplement

Preliminary Example. Use the unit circles and corresponding triangles below to find the exact value of the sine and cosine of the special angles 30° , 45° , and 60° .

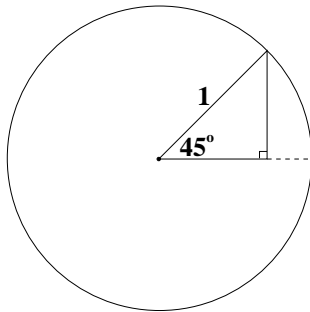


Figure 1.

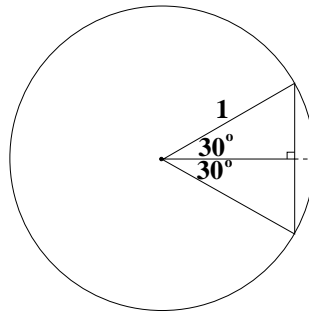


Figure 2.

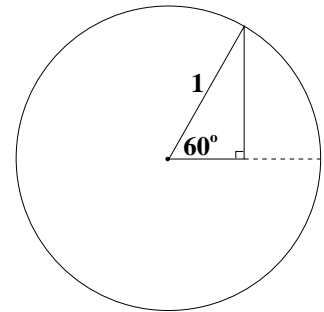
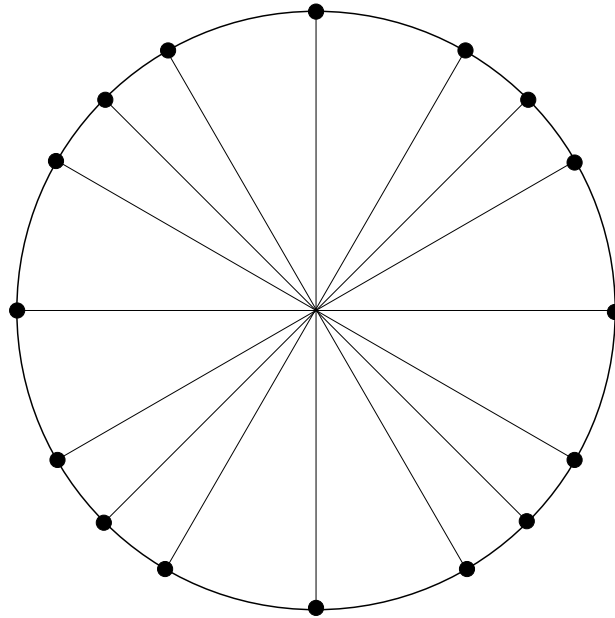


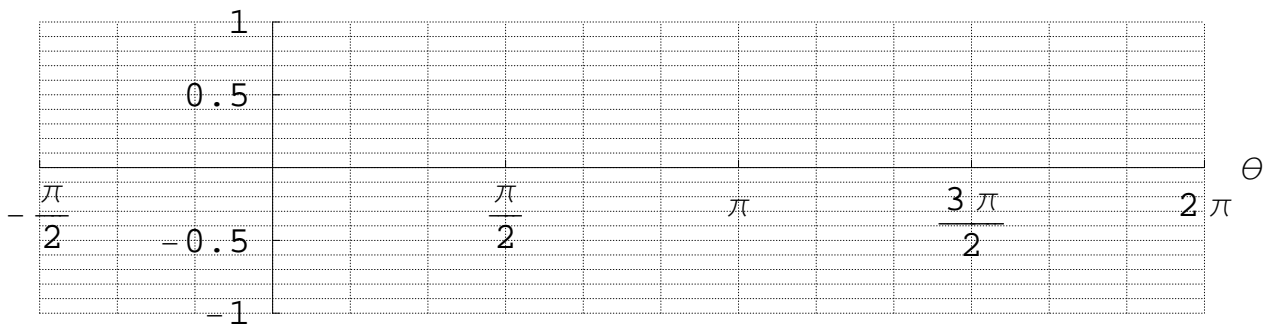
Figure 3.

The Unit Circle



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
cos θ									
sin θ									

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
θ	210°	225°	240°	270°	300°	315°	330°	360°
cos θ								
sin θ								



Sections 6.4 and 6.5 – Sinusoidal Functions

Directions. Make sure that your graphing calculator is set in radian mode.

Function	Effect on $y = \sin x$
$y = 2 \sin x$	
$y = \sin x + 2$	
$y = \sin(x + 2)$	
$y = \sin(2x)$	

B	$y = \sin(Bx)$	Period
1	$y = \sin x$	
2	$y = \sin(2x)$	
4	$y = \sin(4x)$	
1/2	$y = \sin(x/2)$	
B	$y = \sin(Bx)$	

Summary

For the sinusoidal functions $y = A \sin(B(x - h)) + k$ and $y = A \cos(B(x - h)) + k$:

1. Amplitude = _____
2. Period = _____
3. Horizontal Shift = _____
4. Midline: _____

Definition. A function is called *sinusoidal* if it is a transformation of a sine or a cosine function.

Primary Goal in Section 6.5. Find formulas for sinusoidal functions given graphs, tables, or verbal descriptions of the functions.

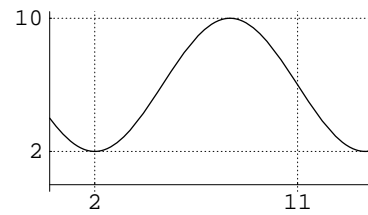
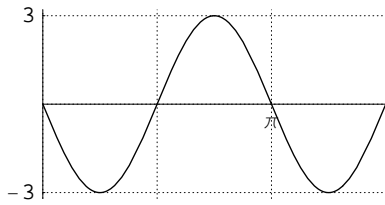
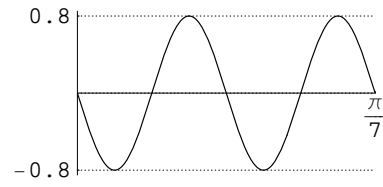
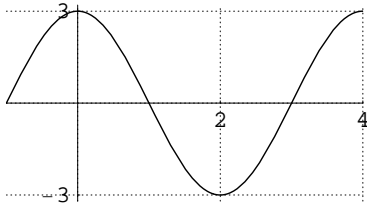
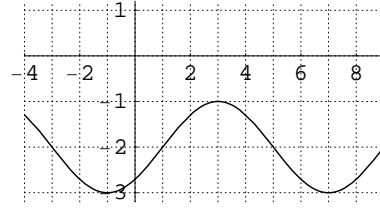
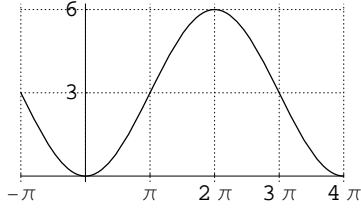
Helpful Hints in Finding Formulas for Sinusoidal Functions

1. If selected starting point occurs at the midline of the graph, use the sine function.
2. If selected starting point occurs at the maximum or minimum value of the graph, use the cosine function.
3. Changing the sign of the constant “A” reflects the graph of a sinusoidal function about its midline.

Example 1. Let $y = 2 \sin(2x - \pi) + 2$. Find the amplitude, period, midline, and horizontal shift of this function.

Examples and Exercises

1. Find a possible formula for each of the following sinusoidal functions.



2. For each of the following, find the amplitude, the period, the horizontal shift, and the midline.

(a) $y = 2 \cos(\pi x + \frac{2\pi}{3}) - 1$

(b) $y = 3 - \sin(2x - 7\pi)$

3. A population of animals oscillates annually from a low of 1300 on January 1st to a high of 2200 on July 1st, and back to a low of 1300 on the following January. Assume that the population is well-approximated by a sine or a cosine function.

(a) Find a formula for the population, P , as a function of time, t . Let t represent the number of months after January 1st. (**Hint.** First, make a rough sketch of the population, and use the sketch to find the amplitude, period, and midline.)

(b) Estimate the animal population on May 15th.

(c) On what dates will the animal population be halfway between the maximum and the minimum populations?

Section 6.6 – Other Trigonometric Functions

$$\cos \theta =$$

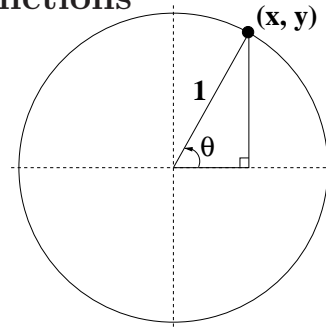
$$\sin \theta =$$

$$\tan \theta =$$

$$\sec \theta =$$

$$\cot \theta =$$

$$\csc \theta =$$



Example 1. Suppose that $\cos \theta = \frac{2}{5}$ and that θ is in the 4th quadrant. Find $\sin \theta$ and $\tan \theta$ exactly.

Example 2. Find exact values for each of the following:

(a) $\tan\left(\frac{\pi}{6}\right)$

(b) $\tan\left(\frac{\pi}{4}\right)$

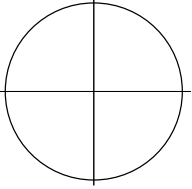
(c) $\tan\left(\frac{\pi}{2}\right)$

Reference Angles

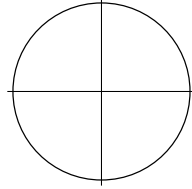
Definition. The *reference angle* associated with an angle θ is the acute angle (having positive measure) formed by the x -axis and the terminal side of the angle θ .

Example. For each of the following angles, sketch the angle and find the reference angle.

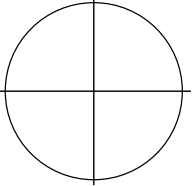
(1) $\theta = 300^\circ$



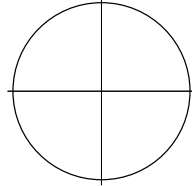
(2) $\theta = \frac{4\pi}{3}$



(3) $\theta = 135^\circ$



(4) $\theta = \frac{7\pi}{6}$



Key Fact. If θ is any angle and θ' is the reference angle, then

$$\sin \theta' = \pm \sin \theta$$

$$\tan \theta' = \pm \tan \theta$$

$$\sec \theta' = \pm \sec \theta$$

$$\cos \theta' = \pm \cos \theta$$

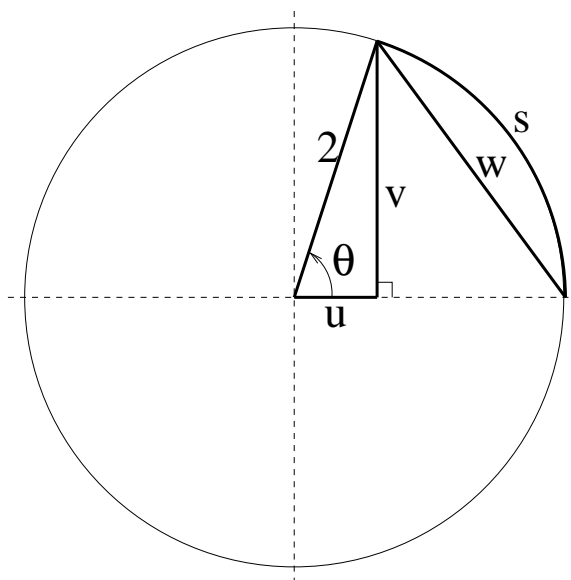
$$\csc \theta' = \pm \csc \theta$$

$$\cot \theta' = \pm \cot \theta,$$

where the correct sign must be chosen based on the quadrant of the angle θ .

Exercise. Return to the previous example and find the exact value of the sine and the cosine of each angle.

3. Given to the right is a circle of radius 2 feet (not drawn to scale). The length of the circular arc s is 2.6 feet. Find the lengths of the segments labeled u , v , and w . Give all answers rounded to the nearest 0.001.



Section 6.7 – Inverse Trigonometric Functions

Preliminary Idea.

$\sin(\pi/6) = 1/2$ means the same thing as _____.

Definition.

1. $\sin^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x .
2. $\tan^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x .
3. $\cos^{-1} x$ is the angle between 0 and π whose cosine is x .

Note. “ $\sin^{-1} x$ ”, “ $\cos^{-1} x$ ”, and “ $\tan^{-1} x$ ” can also be written as “ $\arcsin x$ ”, “ $\arccos x$ ”, and “ $\arctan x$ ”, respectively.

Example 1. Calculate each of the following exactly.

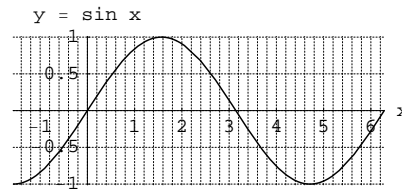
1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \underline{\hspace{2cm}}$

3. $\tan^{-1}(\sqrt{3}) = \underline{\hspace{2cm}}$

2. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \underline{\hspace{2cm}}$

4. $\sin^{-1}(-1) = \underline{\hspace{2cm}}$

Example 2. Use the graph to estimate, to the nearest 0.1, all solutions to the equation $\sin x = -\frac{1}{2}$ that lie between 0 and 2π . Then, find the solutions exactly.



Examples and Exercises

1. Solve each of the following trigonometric equations, giving all solutions between 0 and 2π . Give **exact** answers whenever possible.

(a) $\sin \theta = \frac{\sqrt{3}}{2}$

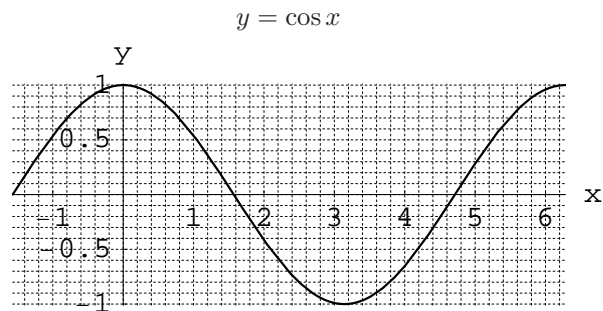
(b) $\tan \theta = -0.3$

(c) $\cos \theta = -\frac{1}{2}$

(d) $\sin \theta = 0.7$

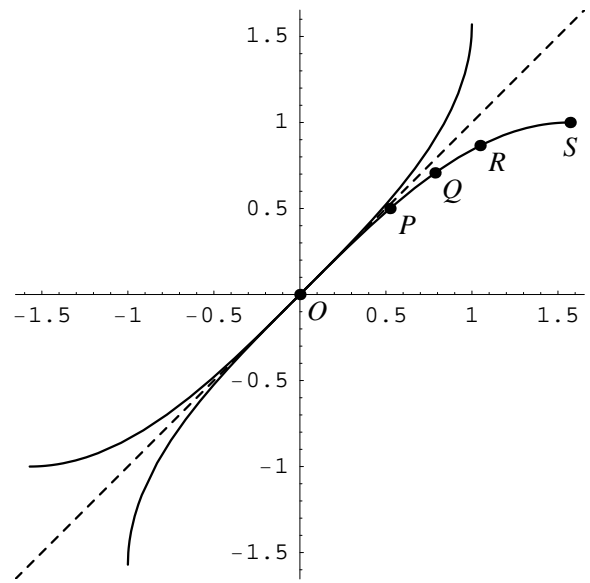
2. Find all solutions to $2 \sin x \cos x + \cos x = 0$ that lie between 0 and 2π . Give your answers **exactly**.

3. Use the graph to the right to estimate the solutions to the equation $\cos x = 0.8$ that lie between 0 and 2π . Then, use reference angles to find more accurate estimates of your solutions.



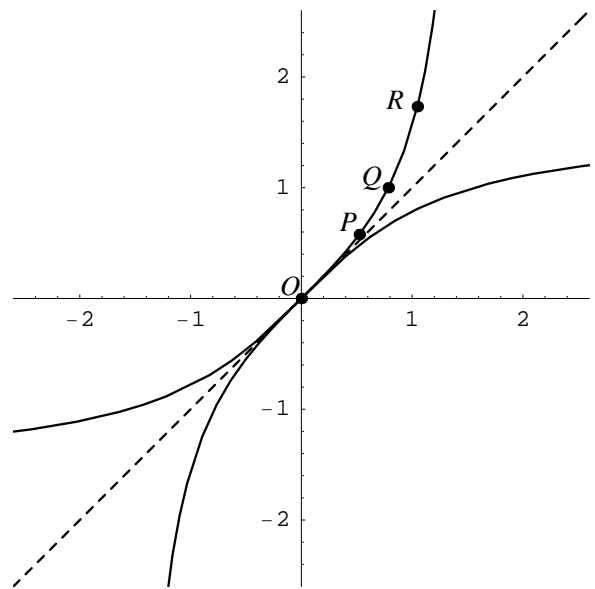
x	$\sin x$
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

x	$\sin^{-1} x$



x	$\tan x$
0	0
$\pi/6$	$\sqrt{3}/3$
$\pi/4$	1
$\pi/3$	$\sqrt{3}$
$\pi/2$	undefined

x	$\tan^{-1} x$



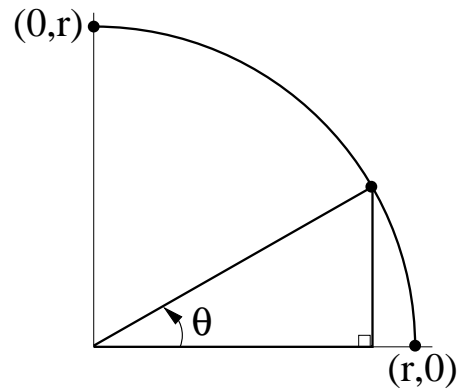
Section 7.1 – The Laws of Sines and Cosines

Right Triangles

$$\sin \theta =$$

$$\cos \theta =$$

$$\tan \theta =$$

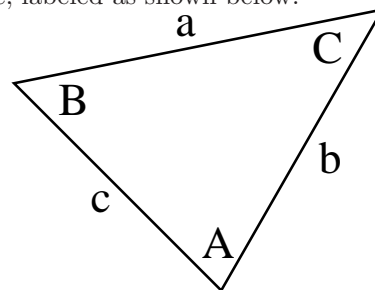


Warmup example. A kite flyer wondered how high her kite was flying. She used a protractor to measure an angle of 40° from level ground to the kite string. If she used a full 100-yard spool of string, how high is the kite?

General Triangles: The following formulas hold for *any* triangle, labeled as shown below.

Law of Sines:

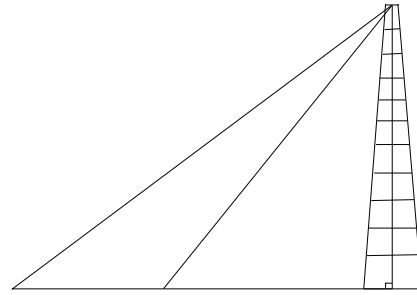
Law of Cosines:



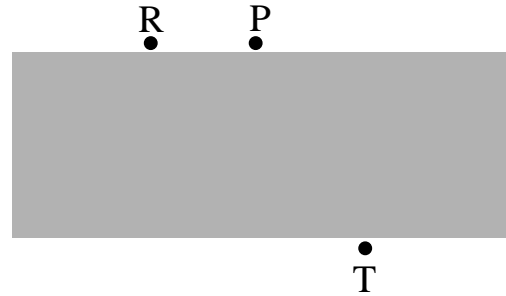
General Rule. The Law of Cosines can be used when 2 sides of a triangle and the angle in between the sides are known.

Example. Find all possible triangles with $a = 3$, $b = 4$, and $A = 35^\circ$.

3. To measure the height of the Eiffel Tower in Paris, a person stands away from the base and measures the angle of elevation to the top of the tower to be 60° . Moving 210 feet closer, the angle of elevation to the top of the tower is 70° . How tall is the Eiffel Tower? (Taken from *Connally, et. al.*)



4. Two points P and T are on opposite sides of a river (see sketch to the right). From P to another point R on the same side is 300 feet. Angles PRT and RPT are found to be 20° and 120° , respectively. (Taken from *Cohen.*)



- (a) Compute the distance from P to T .

- (b) Assuming that the river is reasonably straight, calculate the shortest distance across the river.

Section 7.2 – Using Trigonometric Identities

Preliminary Example. Discuss the difference between the equation (a) $\sin \theta = \cos \theta$ and the equation (b) $\sin(2\theta) = 2 \sin \theta \cos \theta$

Example 1. Are any of the following identities? If so, prove them algebraically.

(a) $\cos\left(\frac{1}{x}\right) = \frac{1}{\cos x}$

(b) $2 \tan x \cos^2 x = \sin(2x)$

Some Trigonometric Identities

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos(2\theta) = 1 - 2 \sin^2 \theta$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos(2\theta) = 2 \cos^2 \theta - 1$$

$$\sin(-\theta) = -\sin \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan(-\theta) = -\tan \theta$$

Example 2. Rewrite the expression $\sec \theta - \cos \theta$ so that your final answer is a product of two trig functions.

Example 3. Solve the equation $2 \sin^2 \theta = 3 \cos \theta + 3$ for $0 \leq \theta \leq 2\pi$.

Examples and Exercises

1. Rewrite each of the following as indicated by the instructions.

Part	Starting Expression	Rewriting Instructions
(a)	$\sin A(\csc A - \sin A)$	Simplify so that your final answer is a single trig function raised to a power, with no fractions.
(b)	$\frac{1 - \cos^2 \theta}{\cos \theta}$	Simplify so that your final answer has no fractions and is a product of two trig functions, with no fractions.
(c)	$\frac{\cos(2t)}{\cos t + \sin t}$	Simplify so that your final answer is a difference of two trig functions with no fractions. (Hint: Use the identity $\cos(2t) = \cos^2 t - \sin^2 t$ and then factor.)

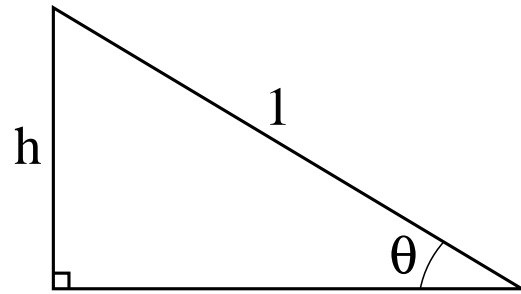
2. Simplify $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta}$ so that your final answer is a constant multiple of just one trig function, with no fractions.

3. By starting with one side and showing that it is equal to the other side, prove the following trigonometric identity:

$$\frac{\sin t}{1 - \cos t} = \frac{1 + \cos t}{\sin t}$$

4. Given the right triangle to the right, write each of the following quantities in terms of h .

Note. Asking you to “write something in terms of h ” is NOT asking you to solve for h . It is asking you to rewrite the quantity that you are given so that h is the only unknown in your answer.



(a) $\sin \theta$	(b) $\cos \theta$
(c) $\cos\left(\frac{\pi}{2} - \theta\right)$	(d) $\sin(2\theta)$
(e) $\cos(\sin^{-1} h)$	

Section 8.1 – Function Composition

The function $h(t) = f(g(t))$ is called the *composition of f with g* . The function h is defined by using the output of the function g as the input of f .

Example 1. Complete the table below.

t	0	1	2	3
$f(t)$	2	3	1	1
$g(t)$	3	1	2	0
$f(g(t))$		3		2
$g(f(t))$		0		1

Example 2. Let $f(x) = x^2 - 1$, $g(x) = \frac{2x^2}{x-1}$, and $p(x) = \sqrt{x}$. Find and simplify each of the following.

- (a) $g(f(x))$ (b) $p(g(x^2))$ (c) $\frac{f(x+h) - f(x)}{h}$

Example 3. For the function $f(x) = (x^3 + 1)^2$, find functions $u(x)$ and $v(x)$ such that $f(x) = u(v(x))$.

Examples and Exercises

1. Given to the right are the graphs of two functions, f and g . Use the graphs to estimate each of the following.

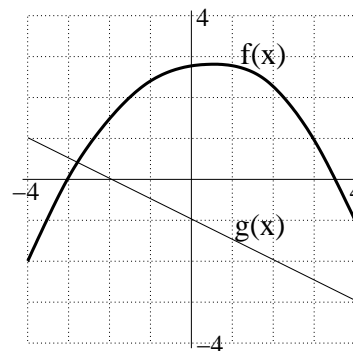
(a) $g(f(0)) = \underline{\hspace{2cm}}$

(b) $f(g(0)) = \underline{\hspace{2cm}}$

(c) $f(g(3)) = \underline{\hspace{2cm}}$

(d) $g(g(4)) = \underline{\hspace{2cm}}$

(e) $f(f(1)) = \underline{\hspace{2cm}}$



2. For each of the following functions $f(x)$, find functions $u(x)$ and $v(x)$ such that $f(x) = u(v(x))$.

(a) $\sqrt{1+x}$

(b) $\sin(x^3 + 1) \cos(x^3 + 1)$

(c) 3^{2x+1}

(d) $\frac{1}{1 + \frac{2}{x}}$

3. Let $f(x) = \frac{1}{1+2x}$.

(a) Solve $f(x+1) = 4$ for x .

(b) Solve $f(x) + 1 = 4$ for x .

(c) Calculate $f(f(x))$ and simplify your answer.

4. For each of the following functions, calculate

$$\frac{f(x+h) - f(x)}{h}$$

and simplify your answers.

(a) $f(x) = x^2 + 2x + 1$

(b) $f(x) = \frac{1}{x}$

(c) $f(x) = 3x + 1$

Section 8.2 – Inverse Functions

Definition. Suppose $Q = f(t)$ is a function with the property that each value of Q determines exactly one value of t . Then f has an *inverse function*, f^{-1} , and

$$f^{-1}(Q) = t \quad \text{if and only if} \quad Q = f(t).$$

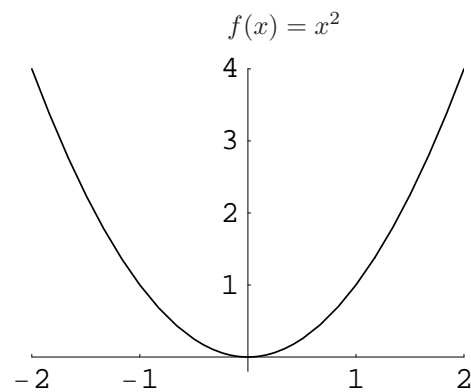
If a function has an inverse, it is said to be *invertible*.

Example 1. Given below are values for a function $Q = f(t)$. Fill in the corresponding table for $t = f^{-1}(Q)$.

t	0	1	2	3	4
$f(t)$	2	5	7	8	11

Q	2	5	7	8	11
$f^{-1}(Q)$					

Question. Does the function $f(x) = x^2$ have an inverse function?



Horizontal Line Test. A function f has an inverse function if and only if the graph of f intersects any horizontal line *at most once*. In other words, if any horizontal line touches the graph of f in more than one place, then f is not invertible.

Example 2. Suppose $B = f(t) = 5(1.04)^t$, where B is the balance in a bank account, in thousands of dollars, after t years.

(a) Find a formula for the inverse function of f .

(b) Compute each of the following and interpret them practically: (i) $f(20)$ (ii) $f^{-1}(20)$

Examples and Exercises

1. Find a formula for the inverse function of each of the following functions.

(a) $f(x) = \frac{x - 1}{x + 1}$

(b) $g(x) = \ln(3 - x)$

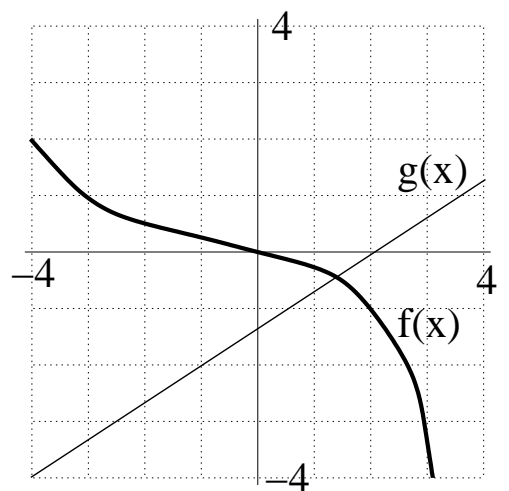
2. Given to the right is the graph of the functions $f(x)$ and $g(x)$. Use the function to estimate each of the following.

(a) $f(2) =$ _____

(b) $f^{-1}(2) =$ _____

(c) $f^{-1}(g(-1)) =$ _____

(d) $g^{-1}(f(3)) =$ _____



(e) Rank the following quantities in order from smallest to largest: $f(1)$, $f(-2)$, $f^{-1}(1)$, $f^{-1}(-2)$, 0

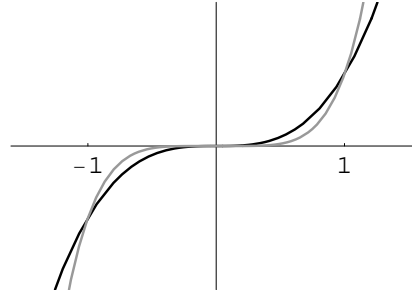
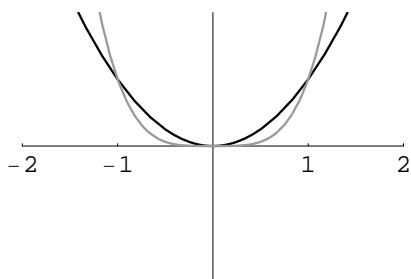
3. Let $f(x) = 10e^{(x-1)/2}$ and $g(x) = 2 \ln x - 2 \ln 10 + 1$. Show that $g(x)$ is the inverse function of $f(x)$.

4. Let $f(t)$ represent the amount of a radioactive substance, in grams, that remains after t hours have passed. Explain the difference between the quantities $f(8)$ and $f^{-1}(8)$ in the context of this problem.

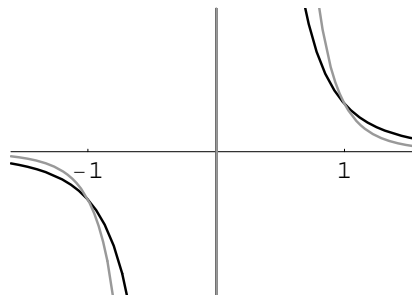
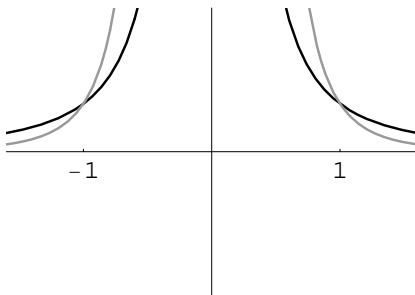
Section 9.1 – Power Functions

Definition. A *power function* is a function of the form $f(x) = kx^p$, where k and p are constants.

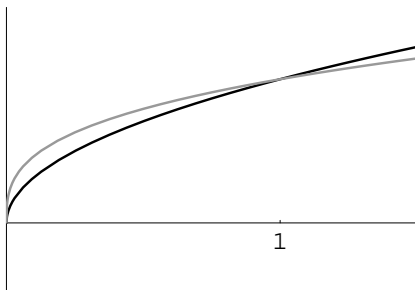
I. Positive Integer Powers. Match the following functions to the appropriate graphs below:
 $y = x^2, y = x^3, y = x^4, y = x^5$



II. Negative Integer Powers. Match the following functions to the appropriate graphs below:
 $y = x^{-2}, y = x^{-3}, y = x^{-4}, y = x^{-5}$



III. Positive Fractional Powers. Match the following functions to the appropriate graphs below:
 $y = x^{1/2}, y = x^{1/3}$



Example 1. Find a formula for the power function that goes through the points $(4, 32)$ and $(\frac{1}{4}, \frac{1}{2})$.

Definition.

1. A quantity y is called *directly proportional* to a power of x if _____, where k and n are constants.
2. A quantity y is called *inversely proportional* to a power of x if _____, where k and n are constants.

Example 2. Write formulas that represent the following statements.

- (a) The pressure, P , of a gas is inversely proportional to its volume, V .
- (b) The work done, W , in stretching a spring is directly proportional to the square of the distance, d , that it is stretched.
- (c) The distance, d , of an object away from a planet is inversely proportional to the square root of the gravitational force, F , that the planet exerts on the object.

Example 3. (Adapted from Connally) The blood circulation time (t) of a mammal is directly proportional to the 4th root of its mass (m). If a hippo having mass 2520 kilograms takes 123 seconds for its blood to circulate, how long will it take for the blood of a lion with body mass 180 kg to circulate?

Examples and Exercises

1. Find a formula for the power function $g(x)$ described by the table of values below. Be as accurate as you can with your rounding.

x	2	3	4	5
$g(x)$	4.5948	7.4744	10.5561	13.7973

2. The pressure, P , exerted by a sample of hydrogen gas is inversely proportional to the volume, V , in the sample. A sample of hydrogen gas in a 2-liter container exerts a pressure of 1.5 atmospheres. How much pressure does the sample of gas exert if the size of the container is cut in half?

Section 9.2/9.3 – Polynomials

Definition. A *polynomial* is a function of the form

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a positive integer and a_0, a_1, \dots, a_n are all constants. The integer n is called the *degree* of the polynomial.

Fact 1. A polynomial of degree n can have at most $n - 1$ “turnaround” points.

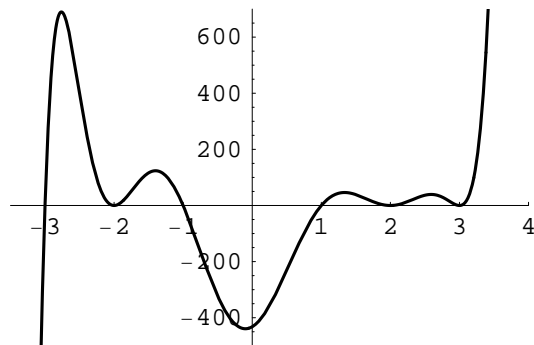
Fact 2. As $x \rightarrow \infty$ and $x \rightarrow -\infty$, the highest power of x “takes over.” (**Note.** The symbol “ \rightarrow ” means “approaches.”)

Fact 3. When a polynomial touches but does not cross the x axis at $x = a$, the factored form of the polynomial will have an even number of $(x - a)$ factors.

Example. Consider the polynomial

$$p(x) = (x + 3)(x + 2)^2(x + 1)(x - 1)(x - 2)^2(x - 3)^2,$$

whose graph is shown to the right.



Definition. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial such that $a_n \neq 0$. Then the number a_n is called the *leading coefficient* of p , and the number a_0 is called the *constant coefficient* of p .

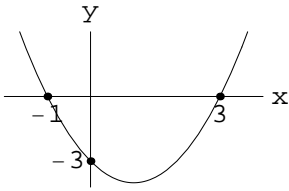
Example. Find the leading coefficient and the constant coefficient of each of the following.

1. $p(x) = 3x^4 - 5x^2 + 6x - 1$
2. $q(x) = x^2(x - 3)$
3. $r(x) = (2x - 3)^2(x + 4)$

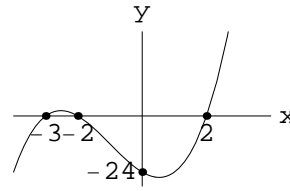
Examples and Exercises

Each of the following gives the graph of a polynomial. Find a possible formula for each polynomial. In some cases, more than one answer is possible.

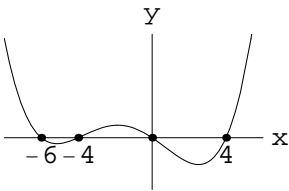
1.



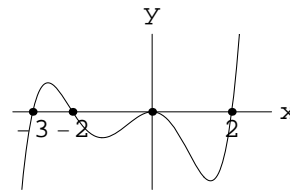
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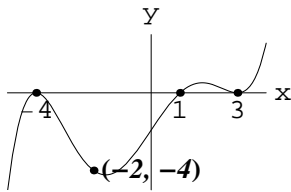
3.



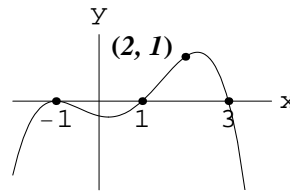
4.



5.



6.



For problems 7 through 10, answer each of the following questions. Use a graphing calculator where appropriate.

- a. How many roots (zeros) does the polynomial have?
- b. How many turning points does the polynomial have?

7. $y = 2x + 3$

8. $y = x^2 - x - 2$

9. $y = x^3 - 2x^2 - x + 2$

10. $y = 5x^2 + 4$

For problems 11 through 15, answer the following questions about the given polynomial:

- a. What is its degree?
- b. What is its leading coefficient?
- c. What is its constant coefficient?
- d. What are the roots of the polynomial? First, give your answer(s) in exact form; then, give decimal approximations if appropriate.

11. $p(x) = x^2 - 3x - 28$

12. $p(x) = 8 - 7x$

13. $p(x) = x(2 + 4x - x^2)$

14. $p(x) = 2x^2 + 4$

15. $p(x) = (x - 3)(x + 5)(x - 37)(2x + 4)x^2$

For problems 16 through 18, answer the following questions about the given polynomial:

- a. What happens to the output values for extremely positive values of x ?
- b. What happens to the output values for extremely negative values of x ?

16. $p(x) = -2x^3 + 6x - 2$

17. $p(x) = 2x - x^2$

18. $p(x) = -x^6 - x - 2$

19. For each of the following, give a formula for a polynomial with the indicated properties.

- a. A sixth degree polynomial with 6 roots.

- b. A sixth degree polynomial with no roots.

Sections 9.4 and 9.5 – Rational Functions

Definition. A *rational function* is a function $r(x)$ of the form $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. In other words, a rational function is a polynomial divided by a polynomial.

Example. Let $f(x) = \frac{3x^2 + 2x - 1}{2x^2 + 1}$. First, fill in the table to the right for the function $f(x)$. Then, sketch a graph of $f(x)$ in the space below.

x	1	10	100	1000	10000
$f(x)$					

Example. Algebraically check each of the following for horizontal asymptotes.

(a) $f(x) = \frac{3x^2 + 2x - 1}{2x^2 + 1}$

(b) $g(x) = \frac{2x + 4}{2x^2 + 1}$

(c) $h(x) = \frac{x^6 + 5x^3 - 2x^2 + 1}{x^4 + 2}$

Finding horizontal asymptotes. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function.

1. If the degree of $p(x)$ equals the degree of $q(x)$, then $f(x)$ has a horizontal asymptote at

$$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}.$$

2. If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is a horizontal asymptote.
 3. If the degree of $p(x)$ is greater than the degree of $q(x)$, then $f(x)$ has no horizontal asymptote.

Example. Let $f(x) = \frac{x-1}{x+2}$. What happens when $x = 1$? What happens when $x = -2$? Graph this function for $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$.

x	-1.9	-1.99	-1.999	-1.9999
$f(x)$				

x	-2.1	-2.01	-2.001	-2.0001
$f(x)$				

Finding vertical asymptotes. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. To find vertical asymptotes, look at places where the denominator $q(x) = 0$.

Examples and Exercises

For each of the following rational functions, find all horizontal and vertical asymptotes (if there are any), all x -intercepts (if there are any), and the y -intercept. Find exact and approximate values when possible.

1. $f(x) = \frac{3x - 4}{7x + 1}$

2. $f(x) = \frac{x^2 + 10x + 24}{x^2 - 2x + 1}$

3. $f(x) = \frac{2x^3 + 1}{x^2 + x}$

4. $f(x) = \frac{(x^2 - 4)(x^2 + 1)}{x^6}$

5. $f(x) = \frac{2x + 1}{6x^2 + 31x - 11}$

6. $f(x) = \frac{2x^2 - c}{(x - c)(3x + d)}$, where c and d are constants, and $c \neq 0$.

7. $f(x) = \frac{1}{x - 3} + \frac{1}{x - 5}$

Hint: First, find a common denominator.

8. $f(x) = \frac{x^5 - 2x^4 - 9x + 18}{8x^3 + 2x^2 - 3x}$

Hint: $x^5 - 2x^4 - 9x + 18 = x^4(x - 2) - 9(x - 2)$

9. $f(x) = \frac{1}{x - 1} + \frac{2}{x + 2} + 3$

Hint: First, find a common denominator.

Show your work for credit!

1. Solve each of the following equations for x , and circle your final answer.

(a) $2(x - 3) = 5x + 2$

(b) $-\frac{2}{3}x = 2\left(\frac{4}{3} - \frac{x}{6}\right)$

2. Find the equation of the line that goes through the points $(-1, 4)$ and $(3, 3)$. Write your final answer in slope intercept form, and circle it.